56. Topological Properties of the Unit Sphere of a Hilbert Space.

By Shizuo Kakutani.

Mathematical Institute, Osaka Imperial University.

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§ 1. The well known fixed point theorem of L. E. J. Brouwer concerning continuous mappings of a finite dimensional closed solid sphere into itself was extended by J. Schauder and A. Tychonoff to the case of an infinite dimensional space in the following way:

Let $K$ be a compact (=bicompact) convex set in a locally convex topological linear space $X$, and let $x' = \varphi(x)$ be a continuous mapping of $K$ into itself. Then there exists a point $x_0 \in K$ such that $x_0 = \varphi(x_0)$.

The purpose of this note is to investigate whether the same or an analogous thing is true for the closed solid unit sphere $K = \{x : \|x\| \leq 1\}$ of a Hilbert space $H$. Since $K$ is compact with respect to the weak topology of $H$, the result quoted above implies that there always exists a fixed point for any weakly continuous mapping of $K$ into itself. Concerning strongly continuous mappings of $K$ into itself, however, there seems to be no published result. In the following lines we shall first show that the fixed point theorem does not hold for strongly continuous mappings of $K$ into itself. In fact, we can even show that there exists a homeomorphism (with respect to the strong topology of $H$) of $K$ onto itself which has no fixed point at all (Theorem 1). This result will then be applied to show that the surface $S = \{x : \|x\| = 1\}$ of $K$ is a retract (in the sense of K. Borsuk) in $K$ (Theorem 2), and further that the identity mapping $x' = x$ is homotopic with the constant mapping $\varphi_0(x) \equiv x_0$ on $S$ (Theorem 3). The paper is concluded with some unsolved problems.

§ 2. Theorem 1. There exists a homeomorphism (with respect to the strong topology) $x' = \varphi(x)$ of the closed solid unit sphere $K = \{x : \|x\| \leq 1\}$ of a Hilbert space $H$ onto itself which has no fixed point.