94. *Normed Rings and Spectral Theorems, II.*

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§ 1. *Introduction.* The purpose of this note is to give an algebraic treatment and version of Fredholm-Riesz-Schauder's theory[1] of completely continuous (c.c.) functional equations with the aide of the theory of normed ring[2]. Our method seems to be suited to obtain the results concerning conjugate equations. We also give a proof to S. Nikolski's extension[3], which is of importance in view of applications, of F-R-S's theory. Lastly we extend the existence theorem of proper values $\neq 0$ for c.c. hermitian operator $\neq 0$ as a corollary of our arguments.

§ 2. *Preliminaries and lemmas.* Let V be a linear operator from a complex Banach space E into E. V is called c.c. if it transforms any bounded set into a compact set. Let $\boldsymbol{R}$ be the commutative ring generated by the c.c. V and the identity operator I, completed by the uniform limit defined by the norm $\|T\| = \sup_{|x|\leq 1} \|T\cdot x\| \cdot \boldsymbol{R}$ is a normed ring with unit I and the norm $\|T\|$, such that any element $T \in \boldsymbol{R}$ may be represented as $T = \lambda I - U$, U being c.c. Let E^* denote the conjugate space $\big($= the space of all the linear functionals f on E with the norm $\|f\| = \sup_{|x|\leq 1} |f(x)|\big)$, then the operator T^* conjugate to T is defined by $T^* \cdot f = g$, $f(T\cdot x) = g(x)$, $f \in E^*$.

Lemma 1. Since[4] $\|T\| = \|T^*\|$, the set $\boldsymbol{R}^*$ of all the operators T^*, $T \in \boldsymbol{R}$, is also a normed ring linear-isomorphic and linear-isometric with $\boldsymbol{R}$ by the correspondence $T \leftrightarrow T^*$.

Lemma 2[5]. $\boldsymbol{R}$ is, as a normed ring, linear homomorphically mapped upon a complex-valued function ring defined on the space $\mathfrak{M}$ of all the maximal ideals $\boldsymbol{M}$ of $\boldsymbol{R}$: $\boldsymbol{R} \ni T \rightarrow T(\boldsymbol{M})$, $I \rightarrow I(\boldsymbol{M}) \equiv 1$ such that $T \equiv T(\boldsymbol{M})I$ (mod. $\boldsymbol{M}$), $\sup_{\boldsymbol{M}} |T(\boldsymbol{M})| = \lim_{n\to\infty} \sqrt[n]{\|T^n\|}$. Moreover T admits inverse T^{-1} in $\boldsymbol{R}$ if and only if $T(\boldsymbol{M}) \neq 0$ on $\mathfrak{M}$.

Lemma 3[6]. Let $I_1 \in \boldsymbol{R}$ be an idempotent viz. $I_1^2 = I_1$, then the

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1) See, for example, S. Banach's book: Théorie des opérations linéaires, Warsaw (1932), 151. Cf. M. Nagumo: Jap. J. of Math., **13** (1936), 6.

2) I. Gelfand: Rec. Math., **9** (1941), 3.

3) C. R. URSS, **16** (1926), 315. Cf. also K. Yosida: Jap. J. Math., **15** (1939), 297.

4) S. Banach: loc. cit., 100.

5) I. Gelfand: loc. cit.

6) Cf. I. Gelfand: loc. cit., 18. For the sake of completeness we will give the proof below. For any $T \in \boldsymbol{R}$ we have $(T - T(\boldsymbol{M})I)I_1 \in \boldsymbol{M} \frown \boldsymbol{R}_1$, since $(T - T(\boldsymbol{M})I) I_1(\boldsymbol{M}) = (T(\boldsymbol{M}) - T(\boldsymbol{M})1)(I_1(\boldsymbol{M})) = 0$, $(T - T(\boldsymbol{M})I)I_1 \cdot I_1 = (T - T(\boldsymbol{M})I)I_1$. Thus $TI_1 = T(\boldsymbol{M})I_1 + (T - T(\boldsymbol{M})I)I_1 \equiv T(\boldsymbol{M})I_1$ (mod. $\boldsymbol{M} \frown \boldsymbol{R}_1$), proving that $\boldsymbol{M}' = \boldsymbol{M} \frown \boldsymbol{R}_1$ is a maximal ideal of $\boldsymbol{R}_1$. Next let $\boldsymbol{M}'$ be a maximal ideal of $\boldsymbol{R}_1$. We will show that there exists a maximal ideal $\boldsymbol{M}$ of $\boldsymbol{R}$ such that $\boldsymbol{M} \geqq \boldsymbol{M}'$, $\boldsymbol{M} \not\ni I_1$. To this purpose consider $\boldsymbol{M} = \boldsymbol{M}' + \boldsymbol{R}(I - I_1)$. That $\boldsymbol{M} \not\ni I_1$ is trivial. Since $T = TI_1 + T(I - I_1)$, $TI_1 \equiv \lambda I_1$ (mod. $\boldsymbol{M}'$ by the lemma 2), we obtain $T \equiv \lambda I_1$ (mod. $\boldsymbol{M}$), proving that $\boldsymbol{M}$ is a maximal ideal of $\boldsymbol{R}$.