127. *On Compact Topological Rings.*

by Hirotada ANZAI.

Mathematical Institute, Osaka Imperial University.

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In this paper we shall study compact or locally compact topological rings, where, by a topological ring, we mean a ring with topology with respect to which the operations $x-y$ and xy are continuous as a function of two variables. We do not assume that the multiplication is commutative. When a topological ring R is observed as an abelian group with respect to addition, R is denoted by G. In § 1, we shall discuss the case when R is compact, by representing R as the ring of endomorphisms of the character group G^* of G. In § 2, we shall give some remarks on locally compact rings by making use of the results obtained in § 1.

§ 1. Let R be a compact topological ring, and G^* the character group of $R(=G)$. The mapping $x \to (\varphi, xa)$, where $x \in G$, $\varphi \in G^*$, and $a \in R$ is fixed, gives rise to a new character $\theta_a \varphi \in G^*$ which is defined by $(\theta_a\varphi, x)=(\varphi, xa)$. It is easy to see that $\varphi \to \theta_a\varphi$ is an endomorphism of G^* into itself. The set of all endomorphisms θ_a of G^*, where a runs through R, is denoted by R^*. Clearly $\theta_{a+b}=\theta_a+\theta_b$ and $\theta_{ab}=\theta_a\theta_b$. Thus $a \to \theta_a$ determines a homomorphism Γ from R onto R^*.

Let us introduce a topology into the ring Θ of all endomorphisms θ of G^*[1]. To this end it suffices to give a system of neighborhoods of the zero endomorphism. We define a neighborhood of zero as follows:

$$V^*_{\varphi_1,\dots,\varphi_n;\,F,\varepsilon}(0)=\Big\{\theta\,\Big|\,|(\theta\varphi_i, x)|<\varepsilon \text{ for all } x\in F,\, i=1,\dots,n\Big\},$$

where $\varphi_i \in G^*, i=1,\dots,n$, F is an arbitrary compact set in G, and $\varepsilon>0$ is an arbitrary positive number. With respect to this topology, Θ is obviously a topological ring. As a subset of Θ, R^* is also topologized. We shall now prove that Γ is continuous as a mapping of R onto R^*. For this purpose let us consider the set

$$A=\Big\{a\,\Big|\,a\in R,\ |(\theta_a\varphi_i, x)|<\varepsilon \text{ for all } x\in F,\ i=1,\dots,n\Big\}$$
$$=\Big\{a\,\Big|\,a\in R,\ |(\varphi_i, xa)|<\varepsilon \text{ for all } x\in F,\ i=1,\dots,n\Big\},$$

where $\varphi_i \in G^*$, $i=1,\dots,n$, F is an arbitrary compact set in G, and $\varepsilon>0$ is an arbitrary positive number. We first note that, for any $\varphi \in G^*$, $\Big\{x\,\Big|\,|(\varphi, x)|<\varepsilon\Big\}$ is an open set in G. Then, by appealing to the following lemma, it is easy to see that A is an open set in G, which implies that Γ is continuous.

Lemma. Let F be a compact set in R, and let U be an open set

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1) S. Kakutani informed the author of the fact that the topological ring Θ had been discussed by M. Abe in his note: Über die Automorphismen der lokalbikompakten abelschen Gruppen, Proc. **15** (1940), 59.