PAPERS COMMUNICATED

1. On the Completion by Cuts of Distributive Lattices.

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"Is the completion by cuts of modular lattices modular? Is that of distributive lattices necessarily distributive? This problem was presented by H. Macneille[1]. G. Birkhoff has listed this problem in his book among unsolved problems[2]. In § 2 we will solve this problem negatively by constructing an example of a distributive lattice, whose completion by cuts is not modular. In § 3 we will give a necessary and sufficient condition for the distributivity of the lattice completed by cuts of a distributive lattice.

1. *Explanation of the problem.*

Let S be a subset of a lattice L, S^+ the set of all upper bounds of S, and S^* the set of all lower bounds of S. We call $\bar{S}=(S^+)^*$ the "normal hull" of S, and S a "normal subset" if and only if it is its own normal hull. If S consists of an element x, then $\bar{x}$ is the set of $y \leqq x$, and $\bar{x}$ is called a "principal" normal subset. All the normal subsets of L, ordered with respect to set inclusion, form a complete lattice $\bar{L}$[3]. All the principal normal subsets form a sublattice isomorphic to L. Our problem is to discuss the distributivity of $\bar{L}$ assuming that L is distributive.

In the discussion of distributivity, the notion of "neutral element" is very important. We define an element a to be neutral if and only if every triple $\{a, x, y\}$ generates a distributive sublattice. The neutral elements of a lattice L constitute a distributive sublattice of L. Thus $\bar{L}$ is distributive if and only if all the elements of $\bar{L}$ are neutral[4].

2. *Example.*

Let L_1, L_2 and L_3 be three simply ordered lattice (i. e. chain) such that

$$L_1\,;\ a_1 > a_2 > \cdots > a_i > \cdots b_j > \cdots > b_2 > b_1$$

$$L_2\,;\ p > q$$

$$L_3\,;\ c_1 > c_2 > \cdots > c_k > \cdots > d_1 > \cdots d_2 > d_1$$

Let L be a sublattice of the direct product $L_1 \times L_2 \times L_3$, consisting of the following elements,

1) H. Macneille, Partially ordered sets, Trans. Amer. Math. Soc., **42** (1937).
2) G. Birkhoff, Lattice theory, 146.
3) loc. cit. 1) or 2).
4) G. Birkhoff, Neutral elements in general lattice, Bull. Amer. Math. Soc., **46** (1940).