18. *Notes on Divergent Series and Integrals.*

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§ 1. The purpose of this paper is to prove the following two theorems:

Theorem 1. Let $x(\omega)$ *and* $y(\omega)$ *be two real-valued non-negative measurable functions defined on the interval* $\Omega=\{\omega \mid 0 \leqq \omega \leqq 1\}$ *of real numbers which are not necessarily integrable on* Ω. *If*

$$(1) \qquad \int_E y(\omega)d\omega < \infty \qquad \textit{implies} \qquad \int_E x(\omega)d\omega < \infty$$

for any measurable subset E *of* Ω, *then there exist a constant* K *and a real-valued non-negative measurable function* $z(\omega)$ *defined and integrable on* Ω *such that*

$$(2) \qquad x(\omega) \leqq Ky(\omega)+z(\omega) \qquad \textit{for any} \qquad \omega \in \Omega .$$

Theorem 2. Let $\{a_n \mid n=1, 2, \dots\}$ *and* $\{b_n \mid n=1, 2, \dots\}$ *be two sequences of real non-negative numbers not greater than* 1, *for which the series* $\sum_{n=1}^{\infty} a_n$ *and* $\sum_{n=1}^{\infty} b_n$ *are not necessarily convergent. If*

$$(3) \qquad \sum_{k=1}^{\infty} b_{n_k} < \infty \qquad \textit{implies} \qquad \sum_{k=1}^{\infty} a_{n_k} < \infty$$

for any subsequence $\{n_k \mid k=1, 2, \dots\}$ *of the sequence* $\{n \mid n=1, 2, \dots\}$ *of all integers, then there exist a constant* K *and a sequence* $\{c_n \mid n=1, 2, \dots\}$ *of real non-negative numbers, for which the series* $\sum_{n=1}^{\infty} c_n$ *is convergent, such that*

$$(4) \qquad a_n \leqq Kb_n+c_n \qquad \textit{for} \qquad n=1, 2, \dots$$

The proof of these theorems will be given in § 3.

§ 2. Let Ω be an arbitrary set and let $\mathfrak{B}=\{E\}$ be a Borel field of subsets E of Ω. Let further $\varphi(E)$ be a countably additive measure defined on $\mathfrak{B}$. We admit the value $+\infty$ for $\varphi(E)$; but in case $\varphi(\Omega) = \infty$, it is assumed that there exists a sequence $\{E_n \mid n=1, 2, \dots\}$ of sets $E_n \in \mathfrak{B}$ such that $\varphi(E_n) < \infty$, $n=1, 2, \dots$ and $\Omega = \bigcup_{n=1}^{\infty} E_n$.

A countably additive measure $\varphi(E)$ defined on $\mathfrak{B}$ is *regular* if, for any $E \in \mathfrak{B}$ with $1 \leqq \varphi(E) \leqq \infty$, there exists an $E' \in \mathfrak{B}$ with $E' \subseteqq E$ and $0 < \varphi(E') \leqq 1$. It is easy to see that, if $\varphi(E)$ is a regular countably additive measure defined on $\mathfrak{B}$, then for any positive number M and for any $E \in \mathfrak{B}$ with $M \leqq \varphi(E) \leqq \infty$, there exists an $E' \in \mathfrak{B}$ with $E' \subseteqq E$ and $M \leqq \varphi(E') \leqq M+1$.

Theorem 3. Let $\varphi(E)$ *and* $\psi(E)$ *be two regular countably additive measures defined on a Borel field* $\mathfrak{B}=\{E\}$ *of subsets* E *of a set* Ω. *If*

$$(5) \qquad \psi(E) < \infty \qquad \text{implies} \qquad \varphi(E) < \infty ,$$

then there exists a constant K *such that*