PAPERS COMMUNICATED

27. Construction of a Non-separable Extension of the Lebesgue Measure Space.

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§ 1. A *measure space* $(\Omega, \mathfrak{B}, m)$ is a triple of a space $\Omega=\{\omega\}$, a Borel field $\mathfrak{B}=\{B\}$ of subsets B of Ω, and a countably additive measure $m(B)$ defined on $\mathfrak{B}$ with $0<m(\Omega)<\infty$. In case Ω is the interval $\{\omega \mid 0 \leqq \omega \leqq 1\}$ of real numbers ω, $\mathfrak{B}$ is the Borel field of all Lebesgue measurable subsets B of Ω, and $m(B)$ is the ordinary Lebesgue measure with $m(\Omega)=1$, $(\Omega, \mathfrak{B}, m)$ is called the *Lebesgue measure space.*

For any measure space $(\Omega, \mathfrak{B}, m)$, let $\mathfrak{p}(\Omega, \mathfrak{B}, m)$ be the smallest cardinal number of a subfamily $\mathfrak{A}$ of $\mathfrak{B}$ with the following property: for any $\varepsilon>0$ and for any $B \in \mathfrak{B}$ there exists an $A \in \mathfrak{A}$ such that $m(B \ominus A)<\varepsilon$, where we denote by $B \ominus A$ the symmetric difference $B \cup A-B \cap A$ of B and A. On the other hand, let $L^2(\Omega, \mathfrak{B}, m)$ be the generalized Hilbert space of all real-valued $\mathfrak{B}$-measurable functions $x(\omega)$ defined on Ω which are square integrable on Ω with $\|x\|=\left(\int_\Omega |x(\omega)|^2 m(d\omega)\right)^{\frac{1}{2}}$ as its norm. Then it is easy to see that $\mathfrak{p}(\Omega, \mathfrak{B}, m)$ is equal with the *dimension* of $L^2(\Omega, \mathfrak{B}, m)$ in case the latter is infinite, where we understand by the dimension of $L^2(\Omega, \mathfrak{B}, m)$ the cardinal number of a complete orthonormal system of $L^2(\Omega, \mathfrak{B}, m)$. We shall call $\mathfrak{p}(\Omega, \mathfrak{B}, m)$ the *character* of a measure space $(\Omega, \mathfrak{B}, m)$.

A measure space $(\Omega, \mathfrak{B}, m)$ is *metrically separable* if $\mathfrak{p}(\Omega, \mathfrak{B}, m) \leqq \aleph_0$. This is equivalent to saying that $L^2(\Omega, \mathfrak{B}, m)$ is separable as a metric space with $d(x, y)=\|x-y\|$ as its distance function. It is clear that the Lebesgue measure space is metrically separable.

A measure space $(\Omega', \mathfrak{B}', m')$ is an *extension* of another measure space $(\Omega, \mathfrak{B}, m)$ if $\Omega'=\Omega$, $\mathfrak{B}' \geqq \mathfrak{B}$ and $m'(B)=m(B)$ on $\mathfrak{B}$. The purpose of this paper is to prove, by constructing an example, the following

Proposition. There exists a metrically non-separable extension of the Lebesgue measure space whose character is $2^{\mathfrak{c}}$.

§ 2. We begin with some lemmas:

Lemma 1. Let S *be an arbitrary set with* $\mathfrak{p}(S)=\mathfrak{c}$[1]. *Then there exists a family* $\mathfrak{S}=\{S_\gamma \mid \gamma \in \Gamma\}$ *of subsets* S_γ *of* S *with the following properties:*

(1) $\mathfrak{p}(\mathfrak{S}) \equiv \mathfrak{p}(\Gamma)=2^{\mathfrak{c}}$,

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1) $\mathfrak{p}(S)$ denotes the cardinal number of a set S,