48. Notes on Fourier Series (XIII). Remarks on the Strong Summability of Fourier Series.

By Tatsuo KAWATA.

Tokyo.

(Comm. by S. KAKEYA, M.I.A., April 12, 1944.)

1. The purpose of this paper is to give some remarks to a paper of Mr. R. Salem[1] concerning the strong summability of Fourier series. He has given sufficient conditions, in terms of the mean modulus of continuity, for strong summability of the Fourier series of a function belonging to L_1. Especially he proved the theorem.

Theorem 1. Let $f(x)$ be an integrable function periodic with period 2π and let $\omega(\delta)$ be its mean modulus of continuity:

$$\omega(\delta)=\max_{0<h\leq\delta}\int_0^{2\pi}|f(x+h)-f(x)|\,dx\,. \tag{1}$$

If $\omega(\delta)=O(1/|\log\delta|^{1+\varepsilon})$, $\varepsilon>0$, then the series

$$\sum_{n=1}^{\infty}\frac{|S_n(x)-\theta_n(x)|}{n} \tag{2}$$

is convergent almost everywhere, where

$$\theta_n(x)=\frac{1}{2}\Big[S_n\Big(x+\frac{\pi}{2n}\Big)+S_n\Big(x-\frac{\pi}{2n}\Big)\Big] \tag{3}$$

and $S_n(x)$ is a partial sum of the Fourier series of $f(x)$.

Salem, in its proof, made use of the following well known theorem

$$\int_0^{2\pi}|S_n(x)|^p\,dx\leq A\sec\frac{p\pi}{2}\Big[\int_0^{2\pi}|f(x)|\,dx\Big]^p,\qquad 0<p<1. \tag{4}$$

2. We shall first remark that we can prove the theorem by using the following fact instead of (4):

$$\int_0^{2\pi}\frac{|S_n(x)|}{\log^{1+\eta}\big(2+|S_n(x)|\big)}\,dx\leq A\int_0^{2\pi}|f(x)|\,dx,\qquad \eta>0, \tag{5}$$

where A depends only on η and this is somewhat convenient for the proof in some point of view. (5) was proved in my previous paper[2] and is an easy consequence of a theorem due to E. T. Titchmarsh concerning the conjugate function. Since $S_n(x)-\theta_n(x)$ is the n-th partial sum of a function $f(x)-\frac{1}{2}f\Big(x+\frac{\pi}{2n}\Big)-\frac{1}{2}f\Big(x-\frac{\pi}{2n}\Big)$, we have

---

1) R. Salem, Sur la convergence en moyenne des séries de Fourier. Comptes Rendus, Paris. t. **208** (1939), pp. 70–72.

2) T. Takahashi (=Kawata). On the conjugate function of an integrable function and Fourier series and Fourier transform, Sci. Rep. Tôhoku Univ. **25** (1936).