47. Notes on Fourier Series (XII). On Fourier Constants.

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1. G. H. Hardy proved[1] that, if a_n are the Fourier sine or cosine coefficients of a function of $L_p(0, 2\pi)$, $p>1$, then their arithmetic means $\frac{1}{n}\sum_1^n a_k$ are also Fourier coefficients of some function of L_p.

In this section, the author considers the another combination of Fourier coefficients instead of arithmetic means. From the well known K. Knopp's inequality

$$\sum_{n=1}^{\infty}\left(\sum_{k=n}^{\infty}\frac{a_k}{k}\right)^2 \leqq 4\sum_{n=1}^{\infty}a_n^2, \tag{1}$$

it readily results that if a_n are Fourier coefficients of a function of L_2, then

$$\sum_{k=n}^{\infty}\frac{a_k}{k} \tag{2}$$

are convergent[2] and are Fourier coefficients of a function of L_2. We ask here whether the similar results will hold for a function of L_p, $p \geqq 1$. With regard to this we obtain the following theorem.

Theorem 1. Let $p>1$ and a_n be the Fourier sine coefficients of a function of L_p. Then (2) are the Fourier sine coefficients of some function of L_p.

We have

$$a_k = \frac{2}{\pi}\int_0^{\pi} f(x)\sin kx dx, \qquad f(x) \in L_p(0, \pi), \qquad p>1.$$

Thus

$$\sum_{k=n}^{\infty}\frac{a_k}{k} = \sum_{k=n}^{\infty}\frac{1}{k}\cdot\frac{2}{\pi}\cdot\int_0^{\pi} f(x)\sin kx dx = \frac{2}{\pi}\int_0^{\pi} f(x)\sum_{k=n}^{\infty}\frac{\sin kx}{k}dx,$$

where the change of order of integration and summation is legitimate since the series $\sum(\sin kx)/k$ is boundedly convergent.

Now since

$$\sum_{n=1}^{\infty}\frac{\sin nx}{n} = \frac{\pi-x}{2} \quad (0<x<\pi)$$

we get

1) G. H. Hardy, On some points in the integral calculus 66. The arithmetic mean of Fourier constants, Messenger of Math. 58 (1928-29).

2) It is well known that if a_n are Fourier sine coefficients, then $\sum a_n/n$ is convergent.