45. Projective Parameters on Paths in D. van Dantzig's Projective Space.

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1. J. Haantjes[1] discussed a few years ago the projective geometry of paths with the use of D. van Dantzig's homogeneous curvilinear coordinates[2]. In a generalized projective space H_n referred to a system of van Dantzig's homogeneous curvilinear coordinate (x^λ)[3], if we introduce a projetive connection $\Pi^\lambda_{\mu\nu}$ satisfying the three conditions: (i) $(\partial\Pi^\lambda_{\mu\nu}/\partial x^\omega)x^\omega = -\Pi^\lambda_{\mu\nu}$, (ii) $\Pi^\lambda_{\mu\nu}=\Pi^\lambda_{\nu\mu}$ and (iii) $\Pi^\lambda_{\mu\nu}x^\mu=0$, the equations of paths or of autoparallel curves are

$$\frac{d^2x^\lambda}{dr^2}+\Pi^\lambda_{\mu\nu}\frac{dx^\mu}{dr}\frac{dx^\nu}{dr}=\alpha\frac{dx^\lambda}{dr}+\beta x^\lambda . \tag{1.1}$$

The finite equations of a path being $x^\lambda(r)$, Haantjes introduces two homogeneous coordinates u^a on each path. Then the equations of paths may also be written as

$$x^\lambda = x^\lambda(u^a) , \tag{1.2}$$

where the $x^\lambda(u^a)$ are homogeneous functions of u^a of degree 1, and the differential equations of paths take the form

$$\frac{\partial^2x^\lambda}{\partial u^b\partial u^c}+\Pi^\lambda_{\mu\nu}\frac{\partial x^\mu}{\partial u^b}\frac{\partial x^\nu}{\partial u^c}=\Gamma^a_{bc}\frac{\partial x^\lambda}{\partial u^a} . \tag{1.3}$$

The functions Γ^a_{bc} appearing in (1.3) transform like the coefficients of a projective connection in an H_1, and satisfy the same conditions as (i), (ii) and (iii). Then, J. Haantjes proves that the curvature tensor formed with the Γ^a_{bc} vanishes identically and therefore there exists a coordinate system (u^a) for which Γ^a_{bc} are all zero. This special coordinate system being determined up to linear homogeneous transformations with constant coefficients, the non homogeneous parameter

$$t=u^1/u^0 \tag{1.4}$$

is determined up to linear fractional transformations. Thus, t is a projective parameter defined on each path. J. Haantjes proves then

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1) J. Haantjes: On the projective geometry of paths, Proc. Edinburgh Math. Soc. **5** (1937), 103-115.

2) D. van Dantzig: Theorie des projektiven Zusammenhangs n-dimensionaler Räume, Math. Ann. **106** (1932), 400-454; J. A. Schouten and J. Haantjes: Zur allgemeinen projektiven Differentialgeometrie, Compositio Math. **3** (1935), 1-51.

3) The indices $\begin{cases}\lambda, \mu, \nu, \dots\\ i, j, k, \dots\\ a, b, c, \dots\end{cases}$ take the values $\begin{cases}0, 1, 2, \dots, n\\ 1, 2, 3, \dots, n\\ 0, 1\end{cases}$ respectively.