44. A Screw Line in Hilbert Space and its Application to the Probability Theory*.

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§ 1. A. Kolmogoroff has investigated the spectralization of the screw line in Hilbert space in his paper "*Kurven in Hilbertschen Raum, die gegenüber einer einparametrigen Gruppen von Bewegung invariant sind*"[1], where he has promised to give the complete proofs in another paper. In this note I will show his results, although the proofs may run in the same way as his own. And I will apply the results to the theory of two-dimensional brownian motions.

§ 2. Under a *congruent transformation in a Hilbert space* we understand an isometric mapping from $\mathfrak{H}$ to $\mathfrak{H}$ itself. On account of the Mazur-Ulam's theorem[2] any congruent transformation K is expressible in the form:

$$K\mathfrak{x} = \mathfrak{a} + U\mathfrak{x} \tag{2.1}$$

where $\mathfrak{a}$ is a fixed element in $\mathfrak{H}$ and U is a unitary operator.

Following after Kolmogoroff, we call a curve $\mathfrak{x}(t)$ in $\mathfrak{H}$ as *a screw line* (induced by a $\| \,\|$-continuous one-parameter group $\{K_t\}$ of congruent transformations), if we have $\mathfrak{x}(t) = K_t\mathfrak{x}(0)$ for any t. We have clearly $\mathfrak{x}(t+s) = K_{t+s}\mathfrak{x}(0) = K_sK_t\mathfrak{x}(0) = K_s\mathfrak{x}(t)$. We define the *moment function* $B_{\mathfrak{x}}(t, \tau, \sigma)$ of any curve $\mathfrak{x}(t)$ by

$$B_{\mathfrak{x}}(t, \tau, \sigma) = \big(\mathfrak{x}(t+\tau) - \mathfrak{x}(t), \mathfrak{x}(t+\sigma) - \mathfrak{x}(t)\big). \tag{2.2}$$

Theorem 1. A necessary and sufficient condition that $\mathfrak{x}(t)$ should be a screw line is that $B_{\mathfrak{x}}(t, \tau, \sigma)$ is independent of t and continuous in τ and σ.

Proof. The *necessity* is clear by the identity:

$$B_{\mathfrak{x}}(t, \tau, \sigma) = \big(K_\tau\mathfrak{x}(0) - \mathfrak{x}(0), K_\sigma\mathfrak{x}(0) - \mathfrak{x}(0)\big).$$

Sufficiency. The following proof is essentially due to Mr. K. Yosida. Suppose that $B_{\mathfrak{x}}(t, \tau, \sigma) = B(\tau, \sigma)$, where $B(\tau, \sigma)$ is continuous in τ and σ. Let $\mathfrak{H}_1$ be the linear manifold determined by the set $\mathfrak{x}(t) - \mathfrak{x}(s)$, $-\infty < s, t < \infty$, and $\mathfrak{H}_2$ be $\mathfrak{H} \ominus \bar{\mathfrak{H}}_1$. Since we have

$$\Big\|\textstyle\sum_i a_i\big(\mathfrak{x}(t_i+\tau) - \mathfrak{x}(s_i+\tau)\big)\Big\|^2 \tag{2.3}$$
$$= \textstyle\sum_{ij} a_i\bar{a}_jB(t_i - s_i, t_j - s_j) = \Big\|\sum_i a_i\big(\mathfrak{x}(t_i) - \mathfrak{x}(s_i)\big)\Big\|^2,$$

the following isometric mapping V_τ can be well defined in $\mathfrak{H}_1$:

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1) C. R. (Doklady), 1940, vol. 26, 1. Cf. also Neumann and Schoenberg: Fourier integral and metric geometry, Trans. Amer. Math. Soc. vol. 50, 2, 1941.

2) Cf. S. Banach: Theorie des opérations linéaires, p. 166.