109. *Stochastic Integral.**

By Kiyosi ITÔ.

Mathematical Institute, Nagoya Imperial University.

(Comm. by S. KAKEYA, M.I.A., Oct. 12, 1944.)

1. *Introduction.* Let (Ω, P) be any probability field, and $g(t, \omega)$, $0 \leqq t \leqq 1$, $\omega \in \Omega$, be any *brownian motion*[1] on (Ω, P) i. e. a (real) stochastic differential process with no moving discontinuity such that $\mathscr{E}\big(g(s, \omega)-g(t, \omega)\big)=0$[2] and $\mathscr{E}\big(g(s, \omega)-g(t, \omega)\big)^2=|s-t|$. In this note we shall investigate an integral $\int_0^t f(\tau, \omega)\, d_\tau g(\tau, \omega)$ for any element $f(t, \omega)$ in a functional class S^* which will be defined in § 2; the particular case in which $f(t, \omega)$ does not depend upon ω has already been treated by Paley and Wiener[3].

In § 2 we shall give the definition and prove fundamental properties concerning this integral. In § 3 we shall establish three theorems which give sufficient conditions for integrability. In § 4 we give an example, which will show a somewhat singular property of our integral.

2. *Definition and Properties.* For brevity we define the classes of measurable functions defined on $[0, 1] \times \Omega$: G, $S(t_0, t_1, \ldots, t_n)$, S and S^* respectively as the classes of $f(t, \omega)$ satisfying the corresponding conditions, as follows,

G: $f(\tau, \omega)$, $g(\tau, \omega)$, $0 \leqq \tau \leqq t$, are independent of $g(\sigma, \omega)-g(t, \omega)$, $t \leqq \sigma \leqq 1$, for any t, $g(\tau, \omega)$ being the above mentioned brownian motion,

$S(t_0, t_1, \ldots, t_n)$, $0=t_0<t_1<\cdots<t_n=1$: $f(t, \omega) \in G \wedge L_2([0, 1] \times \Omega)$ and $f(t, \omega)=f(t_{i-1}, \omega)$, $t_{i-1} \leqq t < t_i$, $i=1, 2, \ldots, n$,

S: $f(t, \omega)$ belongs to $S(t_0, \ldots, t_n)$ for a system $t_0, t_1, \cdots, t_n$ which may depend upon $f(t, \omega)$; in other words $S \equiv \cup S(t_0, t_1, \ldots, t_n)$,

S^*: $f(t, \omega) \in G$ and for any ε there exists $h(t, \omega) \in \bar{S}$[4] such that

$$P\{\omega; f(t, \omega)=h(t, \omega) \text{ for any } t\} > 1-\varepsilon.$$

At first for $f(t, \omega) \in S$ we define the stochastic integral $\int_0^t f(\tau, \omega)\, d_\tau g(\tau, \omega)$ (for brevity denote it by $I(t, \omega; f)$) as follows:

---

* The cost of this research has been defrayed from the Scientific Expenditure of the Department of Education.

1) C. P. Lévy: Théorie de l'addition des variable aléatoire, P. 167, 1937, and also J. L. Doob: Stochastic processes depending on a continuous parameter, Trans., Amer. Math. Soc. vol. 42, Theorem 3.9.

2) $\mathscr{E}$ denotes the mathematical expectation, viz. $\mathscr{E} f(\omega)=\int_\Omega f(\omega) P(d\omega)$.

3) R. E. A. G. Paley and N. Wiener, Fourier transforms in the complex domain, Amer. Math. Soc. Coll. Publ. (1934), Chap. IX.

4) $\bar{S}$ means the closure of S with respect to the norm in $L_2([0, 1] \times \Omega)$.