107. On Biorthogonal Systems in Banach Spaces.

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1. Let $\{x_i\}$ be a sequence of elements of a Banach space E and $\{f_i\}$ a sequence of elements of its conjugate space $\bar{E}$, that is, the space of all bounded linear functionals $f(x)$ defined on E, with norm $\|f\| = \underset{\|x\| \leq 1}{\text{l. u. b.}} |f(x)|$.

The system $\{x_i ; f_i\}$ $(i=1, 2, \ldots)$ is called to be *biorthogonal* if

$$f_i(x_j) = \begin{cases} 1 & \text{for } i=j, \\ 0 & \text{for } i \neq j. \end{cases}$$

We denote by X_k the closed linear subspace of E which consists of all linear combinations of terms of the subsequence of $\{x_i\}$ obtained by taking away only one term x_k from $\{x_i\}$ and of all limits of the combinations. The sequence $\{x_i\}$ is said to be *minimal* if $x_k \bar{\in} X_k$ for all k.

S. Kaczmarz and H. Steinhaus[1] have proved the following theorem:

Let $\{x_i\}$ be a sequence of elements of the space $L^{(p)}(p \geqq 1)$. The necessary and sufficient condition that there exists a sequence $\{f_i\}$ of bounded linear functionals defined on $L^{(p)}$ such that the system $\{x_i ; f_i\}$ is biorthogonal is that the sequence $\{x_i\}$ is minimal.

The object of the present paper is to show that the above theorem is valid in the Banach space E and to get the conditions for the existence of $\{x_i\}$ of elements of E such that for a given sequence $\{f_i\}$ of elements of $\bar{E}$ the system $\{x_i ; f_i\}$ is biorthogonal and finally to apply the obtained results to a trigonometrical system.

2. *Theorem 1. Let $\{x_i\}$ be a sequence of elements of E. The necessary and sufficient condition that there exists a sequence $\{f_i\}$ of elements of $\bar{E}$ such that $\{x_i, f_i\}$ is biorthogonal is that $\{x_i\}$ is minimal.*

Proof. Necessity. Suppose that there exists $\{f_i\}$ such that $\{x_i ; f_i\}$ is biorthogonal and $x_1 \in X_1$. Then there are sequences of numbers $\{\gamma_k^{(n)}\}$ $(n=1, 2, \ldots)$ such that $Z_n = \sum_{k=2}^{m_n} \gamma_k^{(n)} x_k$ and $\lim_{n\to\infty} Z_n = x_1$.

Therefore $$\lim_{n\to\infty} f_1(Z_n) = f_1(x_1) = 1.$$

On the other hand $f_1(Z_n) = 0$ for $n=1, 2, \ldots$, thus we have arrived at a contradiction.

Sufficiency. If $\{x_i\}$ is minimal, then $x_1 \bar{\in} X_1$. Since X_1 is a closed linear subspace of E, there exists an $f_1 \in \bar{E}$ such that

1) S. Kaczmarz and H. Steinhaus: Theorie der Orthogonalreihen, Warszawa, 1935, p. 264.