105. On the Reducibility of the Differential Equations in the n-Body Problem.

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It is known that the system of differential equations for the motion of n bodies can be reduced to a system of differential equations of order $6n-12$ from that of order $6n$ by the aid of the Eulerian integrals of the eliminations of the node and of the time. Lie's theory on the contact transformation groups and the function-groups has been applied for carrying out the effective reduction of the order of this system of differential equations[1]. Among others É. Cartan's procedure is the most elegant in employing the theory of integral invariants[2]. In the present note I propose to modify the procedure by avoiding the explicit appearance of time in the treatment[3] and also to discuss the n-body problem in the planar case.

Let, according to Poincaré[4], $x_{3j-2}, x_{3j-1}, x_{3j}$ be the Cartesian coordinates of the j-th body with mass $m_{3j-2}=m_{3j-1}=m_{3j}$, $(j=1,2,\dots,n)$, and $y_{3j-2}, y_{3j-1}, y_{3j}$ be the Cartesian components of the momentum of the j-th body. Then the motion of the n bodies is represented by the following canonical system of differential equations.

$$\frac{dx_i}{dt}=\frac{\partial H}{\partial y_i}, \qquad \frac{dy_i}{dt}=-\frac{\partial H}{\partial x_i}, \qquad (i=1,2,\dots,3n-1,3n),$$

where

$$H=T-U,$$

$$T=\sum_{k=1}^{3n}\frac{1}{2m_k}y_k^2, \qquad U=\sum_{i\neq j}\frac{m_{3i}m_{3j}}{\Delta_{i,j}},$$

$$\Delta_{i,j}^2=(x_{3i-2}-x_{3j-2})^2+(x_{3i-1}-x_{3j-1})^2+(x_{3i}-x_{3j})^2.$$

This system of differential equations admit the infinitesimal transformations:

$$A_0f=\frac{\partial f}{\partial t}, \quad A_1f=\sum_{j=1}^{n}\frac{\partial f}{\partial x_{3j-2}}, \quad A_2f=\sum_{j=1}^{n}\frac{\partial f}{\partial x_{3j-1}}, \quad A_3f=\sum_{j=1}^{n}\frac{\partial f}{\partial x_{3j}},$$

$$A_4f=\sum_{j=1}^{n}\left(-x_{3j}\frac{\partial f}{\partial x_{3j-1}}+x_{3j-1}\frac{\partial f}{\partial x_{3j}}\right), \quad A_5f=\sum_{j=1}^{n}\left(-x_{3j-2}\frac{\partial f}{\partial x_{3j}}+x_{3j}\frac{\partial f}{\partial x_{3j-2}}\right),$$

1) S. Lie, Math. Ann., **8** (1874), 215; Gesammelte Abhandlung, **4** (1929), 1; Goursat, Leçons sur l'intégration des équations différentielles aux dérivées partielles du premier ordre, 1921; Engel-Faber, Die Lie'sche Theorie der partiellen Differentialgleichungen erster Ordnung, 1932; Englund, Sur les méthodes d'intégration de Lie et le problème de la mécanique céleste, Thèse, Uppsala, 1916; Engel, Göttinger Nachrichten, Math.-Phys. Kl., 1916, 270; 1917, 189.

2) E. Cartan, Leçons sur les invariants intégraux, 1922.

3) Y. Hagihara, Comptes Rendus Acad. Sc. Paris, **207** (1938), 390.

4) H. Poincaré, Bulletin Astr., **14** (1897), 53; Acta Mathematica, **21** (1897), 83 Leçons de mécanique céleste, **1** (1905). Chap. I.