130. On Hopf's Ergodic Theorem.

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1. Let E be a measurable set of points in $|z|<1$. We define its hyperbolic measure $\sigma(E)$ by $\sigma(E)=\iint_E \frac{rdrd\theta}{(1-r^2)^2}$ $(z=re^{i\theta})$. Similarly the hyperbolic length $\lambda(C)$ of a rectifiable curve C is defined by $\lambda(C)=\int_C \frac{|dz|}{1-|z|^2}$.

Let G be a Fuchsian group of linear transformations, which make $|z|<1$ invariant and D_0 be its fundamental domain, which contains $z_0=0$ and is bounded by at most enumerably infinite number of orthogonal circles to $|z|=1$, z_n be equivalents of $z_0=0$ and $n(r)$ be the number of z_n in $|z|\leqq r$. For any z in $|z|<1$, we denote its equivalent in D_0 by (z). Let $E(\theta)$ be the set of points $(re^{i\theta})$ in D_0, which are equivalent to points on a radius $z=re^{i\theta}$ $(0\leqq r<1)$. In my former paper[1], I have proved:

Theorem 1. (i) *If* $\sum_{n=0}^{\infty}(1-|z_n|)=\infty$, *then* $E(\theta)$ *is everywhere dense in* D_0 *for almost all* $e^{i\theta}$ *on* $|z|=1$, (ii) *If* $\sum_{n=0}^{\infty}(1-|z_n|)<\infty$, *then* $\lim_{r\to 1}|(re^{i\theta})|=1$ *for almost all* $e^{i\theta}$ *on* $|z|=1$.

Theorem 2. The necessary and sufficient condition that there exists a set e *on* $|z|=1$, *which is invariant by* G *and* $0<me<2\pi$, *is that* $\sum_{n=0}^{\infty}(1-|z_n|)<\infty$.

Theorem 1 (i) is an extension of Myrberg's theorem[2], who assumed that D_0 lies with its boundary entirely in $|z|<1$, in which case, it is easily proved that $\sum_{n=0}^{\infty}(1-|z_n|)=\infty$.

2. Let $\eta_1=e^{i\theta}$, $\eta_2=e^{i\varphi}$ be two points on $|z|=1$, $|w|=1$ respectively. Then the pair (η_1,η_2) can be considered as a point on a torus $\Omega(0\leqq\theta\leqq 2\pi,\ 0\leqq\varphi\leqq 2\pi)$. For any measurable set E on Ω, we define its measure mE by $mE=\iint_E d\theta d\varphi$, so that $m\Omega=4\pi^2$.

Let S be any substitution of G and T: $\eta_1'=S(\eta_1)$, $\eta_2'=S(\eta_2)$, then the totality of T constitutes a group $\mathfrak{G}$, which is isomorphic to G. Hopf proved the theorem[3]:

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1) M. Tsuji: Theory of conformal mapping of a multiply connected domain, III. Jap. Journ. Math. **19** (1944).

2) Myrberg: Ein Satz über die Fuchsschen Gruppen und seine Anwendungen in der Funktionentheorie. Annales Academie Sci. Fennicae. **32** (1929).

3) E. Hopf: Fuchsian groups and ergodic theory. Trans. Amer. Math. Soc. **39** (1936). Ergodentheorie. Berlin (1937).