128. *Mean Concentration Function and the Law of Large Numbers.*

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1. Let $F(x)$ be a probability distribution. Put

$$Q_F(l)=\sup_{-\infty<x<\infty}\{F(x+l+0)-F(x-0)\}\,,\qquad l>0\,. \tag{1}$$

$Q_F(l)$ is called the *maximal concentration function* of $F(x)$ and plays a fundamental rôle in the theory of P. Lévy on the series of independent random variables[1]. Recently, T. Kawata[2] introduced a *mean concentration function*:

$$C_F(l)=\frac{1}{2l}\int_{-\infty}^{\infty}\{F(x+l+0)-F(x-l-0)\}^2dx\,,\qquad l>0\,, \tag{2}$$

and has shown that most of P. Levy's results can be obtained in an analytical way by appealing to this function. It is easy to see, by Plancherel's theorem and Lévy's inversion formula, that $C_F(l)$ can be expressed by Fejér integral:

$$C_F(l)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{\sin lt}{lt}\right)^2|f(t)|^2dt=2\int_0^{2l}\left(1-\frac{x}{2l}\right)d\widetilde{F}(x)\,, \tag{3}$$

where $f(t)$ is the characteristic function of $F(x)$, and $\widetilde{F}(x)=\big(1-F(-x)\big)*F(x)$ is the symmetrized distribution of $F(x)$. $\big(F(x)*G(x)$ denotes the convolution of two distributions $F(x)$ and $G(x)$. Thus $|f(t)|^2$ is the characteristic function of $\widetilde{F}(x)\big)$. In the present paper we propose to adopt a new mean concentration function:

$$\begin{aligned}\psi_F(l)&=l\int_0^{\infty}e^{-lt}|f(t)|^2dt=\int_0^{\infty}e^{-t}\left|f\left(\frac{t}{l}\right)\right|^2dt\\&=2\int_0^{\infty}\frac{l^2}{l^2+x^2}d\widetilde{F}(x)\,,\qquad l>0\,.\end{aligned} \tag{4}$$

This function, based on Poisson integral, well turn out to be a useful tool in some problems in the theory of independent random variables.

2. It is easy to see that $\psi_F(l)$ possesses similar properties as $Q_F(l)$ and $C_F(l)$. $\psi_F(l)$ is non-negative, monotone non-decreasing in l, and

$$\lim_{l\to\infty}\psi_F(l)=1\,, \tag{5}$$

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1) P. Lévy, L'addition des variables aléatoires, Paris, 1937.

2) T. Kawata, The function of the mean concentration function of a chance variable, Duke Math. Journ., **9** (1941). T. Kawata, Tokyo Buturi Gakko Zassi, **50** (1942), 11.