19. *Some Metrical Theorems on Fuchsian Groups.*

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1. Let E be a measurable set in $|z|<1$. We define its hyperbolic measure $\sigma(E)$ by $\sigma(E)=\iint_E \frac{dxdy}{(1-|z|^2)^2}$ $(z=x+iy)$. Let e be a linear set on a rectifiable curve C in $|z|<1$, then its hyperbolic linear measure $\lambda(e)$ is defined by $\lambda(e)=\int_e \frac{|dz|}{1-|z|^2}$.

Let G be a Fuchsian group of linear transformations, which make $|z|<1$ invariant and D_0 be its fundamental domain, containing $z=0$ and z_n be equivalents of $z_0=0$. For any z in $|z|<1$, we denote its equivalent in D_0 by (z). Let $E(\theta)$ be the set of points $(re^{i\theta})$ in D_0, which are equivalent to points on a radius $z=re^{i\theta}(0\leqq r<1)$ of $|z|=1$. In may formar paper[1], I have proved:

Theorem 1. (*i*) *If* $\sum_{n=0}^{\infty}(1-|z_n|)=\infty$, *then* $E(\theta)$ *is everywhere dense in* D_0 *for almost all* $e^{i\theta}$ *on* $|z|=1$, (ii) *If* $\sum_{n=0}^{\infty}(1-|z_n|)<\infty$, *then* $\lim_{r\to 1}|(re^{i\theta})|=1$ *for almost all* $e^{i\theta}$ *on* $|z|=1$.

In this paper, we will prove the following theorem, which is a precision of Theorem 1 (i).

Theorem 2. Suppose that $\sigma(D_0)<\infty$. *Let* $\wedge$ *be a set in* D_0, *which is measurable in Jordan's sense. Let* $g: z=te^{i\theta}(0\leqq t<1)$ *be a radius of* $|z|=1$ *and* l *be a segment* $(0\leqq t\leqq r)$ *on* g *of length* r, *whose hyperbolic length be* L *and* $L(\wedge)$ *be the hyperbolic measure of the set of t-values on* $(0,r)$, *such that* $(te^{i\theta})\in\wedge$. *Then there exists a set* e_0 *of measure zero on a unit circle* $U: |z|=1$, *which does not depend on* $\wedge$, *such that if* $e^{i\theta}\in U-e_0$, *then for any* $\wedge$,

$$\lim_{L\to\infty}\frac{L(\wedge)}{L}=\frac{\sigma(\wedge)}{\sigma(D_0)}. \tag{1}$$

Proof. We consider D_0 as a Riemann manifold F of constant negative curvature with $ds=\frac{|dz|}{1-|z|^2}$ and equivalent points are considered as the same point of F. Let $z=x+iy$ be any point of D_0. We associate a direction φ at z, which makes an angle φ with the real axis. Then the line elements (z,φ) $(z\in D_0,\ 0\leqq\varphi\leqq 2\pi)$ constitute a phase space Ω, which is a product space of D_0 and a unit circle U: $\Omega=D_0\times U$ and the volume element $d\mu$ in Ω is defined by $d\mu=\frac{dxdyd\varphi}{(1-|z|^2)^2}$, so that $\mu(\Omega)=2\pi\sigma(D_0)<\infty$.

1) M. Tsuji: Theory of conformal mapping of a multiply connected domain, III. Jap. Journ. Math. **19** (1944).