New geometric presentations for $\mathrm{Aut}\, G_2(3)$ and $G_2(3)$

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1 Introduction

The purpose of this article is to provide new presentations for the groups $G_2(3)$ and $\mathrm{Aut}\, G_2(3)$. These presentations come from the amalgam of maximal parabolic subgroups arising in the action of $\mathrm{Aut}\, G_2(3)$ on a certain geometry.

The members of this amalgam are the well-known subgroups of $\hat{G} = \mathrm{Aut}\, G_2(3)$ (cf. [ATL]): $\hat{L} = 2^3 \cdot L_3(2) : 2$, $\hat{N} = 2^{1+4}.(S_3 \times S_3)$, and $M = G_2(2)$. Notice that M is fully contained in $G = O^2(\hat{G}) \cong G_2(3)$, while $\hat{L}$ and $\hat{N}$ are not. This explains our hat notation. According to this notation we set $L = \hat{L} \cap G \cong 2^3 \cdot L_3(2)$ and $N = \hat{N} \cap G \cong 2^{1+4}.(3 \times 3).2$.

We choose the subgroups $\hat{L}$ and M so that $D = \hat{L} \cap M$ is a maximal parabolic subgroup in M. Then D has a unique normal subgroup 2^2 (contained in $O_2(L) \cong 2^3$). Let z be an involution from that normal subgroup. We choose $\hat{N} = C_{\hat{G}}(z)$. This uniquely specifies the amalgam $\hat{\mathcal{A}} = \hat{L} \cup \hat{N} \cup M$. Let $e \in O_2(\hat{L}) \setminus O_2(L)$ and set $K = M^e$. Let $\mathcal{B} = L \cup N \cup M \cup K$. Clearly, $\hat{G} = \langle \hat{\mathcal{A}} \rangle$ and $G = \langle \mathcal{B} \rangle$.

Theorem 1. *$\hat{G} = \mathrm{Aut}\, G_2(3)$ is the universal completion of the amalgam $\hat{\mathcal{A}}$.*

As a corollary of this theorem we get our second main result.

Theorem 2. *$G = G_2(3)$ is the universal completion of the amalgam $\mathcal{B}$.*

As we have already mentioned, the amalgam $\hat{\mathcal{A}}$ is the amalgam of maximal parabolics with respect to the action of $\hat{G}$ on a certain geometry $\hat{\Gamma}$. In this sense, Theorem 1 is equivalent, via Tits' Lemma [T] (also cf. [IS], Theorem 1.4.5), to the simple connectedness of the geometry $\hat{\Gamma}$.

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