Total Mean Curvature and Closed Geodesics

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The purpose of this note is to give a proof of the following theorem and to give some easy applications of it and its proof to the extrinsic geometry of convex Zoll surfaces.

In what follows we will use the word *surface* to mean *smooth closed surface*, and the words *strictly convex* to mean *of positive Gaussian curvature*.

Theorem 1 *On a strictly convex surface $\Sigma \subset \mathbb{R}^3$ there exists a closed geodesic whose length is less than or equal to one half the total mean curvature of Σ.*

The proofs and applications are based on a Riemannian version of Gromov's non-squeezing theorem and classical integral geometry.

Given a convex surface $\Sigma \subset \mathbb{R}^3$ and a point q in the unit sphere S^2 we denote by $U_\Sigma(q)$ the perimeter of the orthogonal projection of Σ onto a plane perpendicular to q. We obtain a function U_Σ on the sphere which is clearly continuous, even, and positive. Let us denote the minimum value for this function by u_Σ. The analogue of the non-squeezing theorem we wish to present is the following result.

Lemma 1 *Let $\Sigma \subset \mathbb{R}^3$ be a strictly convex surface. There exists on Σ a closed geodesic whose length is less than or equal to u_Σ.*

The theorem follows from this lemma and the following integral-geometric characterization of the total mean curvature in terms of the average of the perimeter function over the sphere.

Lemma 2 *Let $\Sigma \subset \mathbb{R}^3$ be a strictly convex surface and let $H := \frac{1}{2}(\kappa_1 + \kappa_2)$ be its mean curvature function. We have that*

$$\int_\Sigma H d\Sigma = \frac{1}{2\pi}\int_{S^2} U_\Sigma d\omega \ .$$

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