An alternate proof of Hall's theorem on a conformal mapping inequality

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In this note we give a different and direct proof of the following result of Hall [2], which actually implies the conjecture of Sheil-Small [3]. For details about the related problems we refer to [1, 3].

THEOREM. *Let f be regular for $|z| < 1$ and $f(0) = 0$. Further, let f be starlike of order $1/2$. Then*

$$\int_0^r |f'(\rho e^{i\theta})| d\rho < \frac{\pi}{2} |f(re^{i\theta})|$$

for every $r < 1$ and real θ.

Proof. As in [2, p.125] (see also [1]), to prove our result it suffices to show that

$$J = I(t, \tau) + I(\tau, t) < \pi - 2 \quad \text{for} \quad 0 < t < \tau < \pi \tag{1}$$

where

$$I(t, \tau) = \int_0^1 \frac{2|\sin(t/2)|}{\sqrt{1 - 2\rho\cos t + \rho^2}} \left\{ \frac{1}{\sqrt{1 - 2\rho\cos\tau + \rho^2}} - \frac{1 - \rho\cos\tau}{1 - 2\rho\cos\tau + \rho^2} \right\} d\rho.$$

To evaluate these integrals we define k by

$$k^2 = \frac{\sin^2(\tau/2) - \sin^2(t/2)}{\cos^2(t/2)\sin^2(\tau/2)}$$

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