Further thoughts on a focusing property of the ellipse

J.B. Wilker

1 Introduction

If a billiard ball is fired from one focus of an elliptical table, its trajectory will pass alternately through the two foci of the table and will quickly become indistinguishable from a repeated tracing of the major axis of the table. In 3-dimensions, an ellipsoid of revolution has focusing properties similar to those of an ellipse. Frantz [1] has used this fact, together with geometrical optics, to follow a spherical wave emitted from one focus of a reflecting ellipsoidal cavity. His analysis predicts a growth in the density of energy tracing the major axis of the cavity which is exponential in time or, more precisely, in the number of reflections which have occurred to the wave. This dramatic prediction is mathematically sound but may not be physically correct since one of the caveats of geometrical optics is to avoid singular situations where spherical disturbances converge to a point (See e.g. [2] p.p. 6, 219). In spite of the fact that the energy interpretation of the model may be open to question, it is interesting to pursue the mathematics because it is attractive and because it does admit other interpretations, for example probabilistic ones.

In this paper we shall derive a probability density at all points and a related estimate concerning the concentration of probability along the major axis. Both of these results generalise the earlier one on density in the direction of the major axis. In order to give a full treatment of the 2-dimensional situation, which is interesting in its own right, we base our approach on the return map to a small circle centred at the first focus. This map turns out to be a Möbius transformation (of hyperbolic

Received by the editors May 1994
Communicated by J. Thas
AMS Mathematics Subject Classification : 78A05, 51N25.
Keywords : reflecting ellipsoidal cavity, return map, Möbius transformation.