Finite geometry for a generation

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Dedicated to J. A. Thas on his fiftieth birthday

There are a number of results concerning the generation of a collineation group by two of its elements. A. A. Albert and J. Thompson [1] were the first to exhibit two elements generating the little projective group $\mathrm{PSL}(d,q)$ of $\mathrm{PG}(d-1,q)$ (for each d and q). According to a theorem of W. M. Kantor and A. Lubotzky [8], "almost every" pair of its elements generates $\mathrm{PSL}(d,q)$ as $qd \to \infty$ (asymptotically precise bounds on this probability are obtained in W. M. Kantor [7]). Given $1 \neq g \in \mathrm{PSL}(d,q)$, the probability that $h \in \mathrm{PSL}(d,q)$ satisfies $\langle g,h\rangle = \mathrm{PSL}(d,q)$ was studied by R. M. Guralnick, W. M. Kantor and J. Saxl [3], and its behavior was found to depend on how $qd \to \infty$. Yet another variation that has been proposed is "$1\frac{1}{2}$"-generation: if $1 \neq g \in \mathrm{PSL}(d,q)$ then *some* $h \in \mathrm{PSL}(d,q)$ satisfies $\langle g,h\rangle = \mathrm{PSL}(d,q)$. This note concerns a stronger version of this notion:

Theorem. *For any $d \geq 4$ and any q, there is a conjugacy class $\mathcal{C}$ of cyclic subgroups of $\mathrm{PSL}(d,q)$ such that, if $1 \neq g \in \mathrm{PSL}(d,q)$, then $\langle g,C\rangle = \mathrm{PSL}(d,q)$ for more than $\left(1-\frac{1}{q}-\frac{1}{q^{d-1}}\right)^2|\mathcal{C}|$ elements $C \in \mathcal{C}$. In particular, there are more than $0.4|\mathcal{C}|$ such elements if $q > 2$.*

While this does not look at all like a geometric theorem, the proof is entirely geometric. The same type of result holds when $d = 2$ or 3 (and is easy), as well as for all the classical groups. The proof by W. M. Kantor [4] for the latter groups is still reasonably geometric, but is harder than the situation of the theorem.

Let V be the vector space underlying $\mathrm{PG}(d-1,q)$. The following is a simple observation concerning the set $\mathrm{Fix}(g)$ of fixed points (in $\mathrm{PG}(d-1,q)$) of a collineation g:

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