On the Structure of the Group of Multiplicative Arithmetical Functions

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Abstract

We analyze the structure of the group $\mathbf{F_0}, \star$ of non-zero multiplicative arithmetical functions, where $\star$ is the usual Dirichlet product. In particular, we prove that $\mathbf{F_0}, \star$ is isomorphic to a complete direct product of certain subgroups of the multiplicative group of infinite upper-triangular matrices. We also show that the group $\mathbf{F_0}, \star$ is divisible.

1 Introduction

An **arithmetical function**, i.e. a function $f : \mathbb{N}_0 \to \mathbb{R}$, is called **multiplicative** if $f(mn) = f(m)f(n)$ whenever $(m, n) = 1$. The Euler function ϕ and the Moebius function μ are classical examples of multiplicative functions. The arithmetical functions $\mathbf{0}$ and $\mathbf{I}$ defined for every $n \in \mathbb{N}_0$ by $\mathbf{0}(n) = 0$, $\mathbf{I}(n) = 0$ or 1 according as $n \neq 1$ or $n = 1$, are trivially multiplicative.

Let $\mathbf{F_0}$ denote the set of all multiplicative functions different from $\mathbf{0}$. Clearly, $f(1) = 1$ for every $f \in \mathbf{F_0}$. The **Dirichlet product** (or **convolution**) of two arithmetical functions f and g is defined as follows: for every $n \in \mathbb{N}_0$,

$$(f \star g)(n) := \sum_{d|n} f(d)g(\frac{n}{d}).$$

For any given prime p, we will consider the following subset of $\mathbf{F_0}$:

$$\mathbf{F}^p = \{f \in \mathbf{F}_0 : f(n) = 0 \text{ for every } n > 1 \text{ not divisible by } p\}$$

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