On the Linkage of Quaternion Algebras

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For quaternion division algebras B, C over a field F (of any characteristic), a well-known theorem of Albert [1] and Sah [6] states that the following conditions are equivalent:

(1) *$B \otimes_F C$ is not a division algebra;*
(2) *B and C have a common quadratic splitting field;*
(3) *some quadratic field extension of F can be embedded (over F) in both B and C.*

In the case where F has characteristic 2, there is a further refinement of this theorem, due to Draxl, which states that the above conditions are also equivalent to[1]:

(4) *B and C have a common separable quadratic splitting field;*
(5) *some quadratic separable field extension of F can be embedded in both B and C.*

Draxl's original proof in [3] was not easy (for me) to follow. Subsequent proofs of the equivalence of (1)–(5) using more advanced tools (respectively, algebraic geometry and the theory of Clifford algebras) appeared in this Bulletin in Tits [7] and Knus [5]. While teaching a course in the theory of division rings, I stumbled upon a short and completely elementary proof of Draxl's part of the above theorems. This proof is recorded below in order to make Draxl's result more easily accessible to non-experts. It has also been known for some time that (4) and (5) are no longer equivalent to (1)–(3) if the word "separable" is replaced by "inseparable". This will

[1]Of course, (4), (5) are also equivalent to (1)–(3) in case $\mathrm{char}(F) \neq 2$, since all quadratic extensions of F are separable in that case. But this would hardly qualify as a "refinement".

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