Convexity and polynomial equations in Banach spaces with the Radon-Nikodym property

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Abstract

We define and characterize a new class of weakly compact subsets of spaces of integrable functions related to a new definition of polynomials in such spaces. We study the algebraic structure of ring of the set of all these polynomials and relate it to geometric and topological properties of subsets of $L_1(\mu)$. We use these results to study relative compactness and convexity of subsets of Banach spaces with the Radon-Nikodym property. An extension of the Uhl theorem about the range of a vector measure is obtained.

1 Introduction and notation.

In this paper we propose new tools for the study of the geometric and topological properties of the range of the integral operator defined by a (countably additive) vector measure of bounded variation. Let (Ω, Σ, μ) be a finite measure space and let $S(\mu)$ be the normed space of all the simple functions of $L_1(\Omega, \Sigma, \mu)$. Consider the (linear) space of polynomials $R[x]$. The procedure that we use consists on the definition of a ring structure with the elements of the tensor product $R[x] \otimes S(\mu)$. We call (Σ, μ)-polynomials the elements of this ring. If $\mathbb{P}$ is a (Σ, μ)-polynomial, we define a polynomial equation $\mathbb{P} = 0$ and find the set of its solutions in the space $L_1(\Omega, \Sigma, \mu)$, i.e., the set of the functions that satisfy $\mathbb{P}(f) = 0$ μ-almost everywhere when we substitute the variable x by the function f in $\mathbb{P}$. The main idea of this paper is to relate the properties of the ring $R[x] \otimes S(\mu)$ to the geometric and topological

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