A note on Serre's condition for orientability of fibre bundles

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Let $F \xrightarrow{i} E \longrightarrow B$ be a fibre bundle with fibre F, and let R be a field. The fibre of the fibre bundle $F \to E \longrightarrow B$ is said to be totally non-homologous to zero in E with respect to R (see e.g. Leray [8] or Mimura, Toda [9]) if the fibre inclusion induces an epimorphism, $i^* : H^*(E; R) \to H^*(F; R)$, in cohomology with coefficients in R. Note (see e.g. [9]) that if B is path connected and F is totally non-homologous to zero with respect to R, then the system of local coefficient rings $\underline{H^*(F; R)}$ is trivial or, in other words, the fibre bundle is R-orientable. In this note, we restrict ourselves to smooth fibre bundles $F \to E \longrightarrow B$ with E and F closed connected manifolds.

If F is totally non-homologous to zero with respect to R, then (see e.g. [9, Chap. 3]) the Serre spectral sequence of the fibration $F \to E \xrightarrow{p} B$ collapses and the Leray-Hirsch theorem applies: $H^*(E; R)$ is free as an $\mathrm{Im}(p^*)$-module with a basis $\{e_\alpha\}$ such that $\{i^*(e_\alpha)\}$ is a homogeneous basis of $H^*(F; R)$ as an R-vector space.

This is one of the reasons why fibre bundles with fibre totally non-homologous to zero are very useful in many situations in topology. For instance, they can be traced behind the answer (given by Korbaš in [3]; see also Sankaran [10]) to the question of when a real flag manifold $\mathbb{R}F(n_1 + \cdots + n_q) := O(n_1 + \cdots + n_q)/O(n_1) \times \cdots \times O(n_q)$ with $q \geq 3$ possesses an almost complex structure. Or a very recent example: Korbaš and Lörinc in [7] succeeded in finding the $\mathbb{Z}_2$-cohomology cup-length and Lyusternik-Shnirel'man category of several infinite families of the real flag manifolds basically using the fact that the manifold $\mathbb{R}F(n_1 + \cdots + n_q)$ can be expressed as the total space of a fibre bundle with fibre totally non-homologous to zero with respect to $\mathbb{Z}_2$. Indeed, an important rôle in their approach is played by a theorem (cf. Horanská, Korbaš [1, Lemma, p. 25]) which can be stated as follows: If for a

Received by the editors March 2005.
Communicated by Y. Félix.
2000 *Mathematics Subject Classification :* Primary 57R22; Secondary 55R10 55R20 57R20.
Key words and phrases : Fibre bundle; flag manifold; fibre totally non-homologous to zero; Serre's orientability condition.