Pappus-Guldin theorems for weighted motions

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1 Introduction

Pappus of Alexandria (fl. c. 300-c. 350), who is regarded as one of the last great mathematicians of the Hellenistic Age, formulated, in the introduction to book VII of his mathematical collections, a rule to determine the volume (resp. area) of a domain in $\mathbb{R}^3$ (resp. surface) generated by rotation of a plane domain D_0 around an axis in its plane: multiply the area (resp. perimeter) of D_0 by the length of the circle generated by the center of mass of D_0 (resp. the center of mass of the boundary of D_0). This rule was rediscovered in 1641 by P. Guldin and was also proved by several mathematicians of the 17th century, such as Kepler and Cavalieri. For that reason, the rule is mainly known as the Pappus-Guldin theorem. In [2] and [16] one can find two reasons which explain why the mathematicians of the early 17th century did not know the Pappus rule. In any case, both articles of the History of Mathematics conclude that P. Guldin did not plagiarize Pappus.

In 1969 A. W. Goodman and G. Goodman [8] developed a generalization of the Pappus-Guldin theorem for domains D generated by moving a plane region D_0 around an arbitrary space curve c. When the center of mass of D_0 is on c, they obtained a Pappus type formula

$$Volume(D) = Area(D_0) \times Length(c). \tag{1.1}$$

Formulas for the area of the surface C generated by moving a plane curve C_0 around c are also considered in [8] but the authors obtain a generalization of the Pappus-Guldin theorem only for plane curves c and 'natural motions' of a plane curve C_0 along c. One year later, L. E. Pursell [15] and H. Flanders [7] supplement the results

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