Monotonous stability for neutral fixed points

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Abstract

We give subtle, simple and precise results about the convergence or the divergence of the sequence (x_n), where $x_j = f(x_{j-1})$ for every integer j, when the initial element x_0 is in the neighbourhood of a neutral fixed point, i.e. a point x^* such that $f(x^*) = x^*$ with $|f'(x^*)| = 1$ (where f is a C^∞ function defined on a subset of $\mathbb{R}$).

1 Introduction

Throughout this paper, we consider a C^∞ function f defined on a subset $S =$ *dom* f of $\mathbb{R}$ and a *fixed point* x^* for f, i.e. a point x^* which will be supposed in the interior of S and such that $f(x^*) = x^*$.

Given a point $x_0 \in S$, we define the *orbit* of x_0 under f to be the infinite sequence of points $x_0, x_1, x_2, \ldots$, where $x_0 = f^0(x_0)$, $x_1 = f(x_0) = f^1(x_0)$, $x_2 = f(x_1) = f^2(x_0), \ldots, x_{n+1} = f(x_n) = f^{n+1}(x_0), \ldots$: the point x_0 is called the *seed* of this orbit which will be denoted by $\mathcal{O}(f; x_0)$ [3, 4].

The aim of this note is to very simply study the asymptotic behavior (i.e. the convergence or divergence) of an orbit the seed of which is in a suitable neighbourhood of a fixed point.

The situation is clear and well-known when x^* is *hyperbolic*, i.e. when $|f'(x^*)| \neq 1$ [3]. Indeed, if $|f'(x^*)| < 1$, then x^* is *stable* or *attracting*; this means that there exists an open interval I which contains x^* and such that $f(I) \subset I$ and $\lim_{n\to\infty} f^n(x) = x^*$ for every $x \in I$ [3, p. 43] [7, p. 45]. Moreover, if $|f'(x^*)| > 1$, then x^* is *unstable* or *repelling*; this means that there exists an open interval I which contains x^* and for which the following condition is satisfied : if $x \in I \setminus \{x^*\}$, there exists an integer $n > 0$ such that $f^n(x) \notin I$ [3, p. 44] [6, p. 20].

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