Some congruences concerning the Bell numbers

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Abstract

In this Note we give elementary proofs – based on umbral calculus – of the most fundamental congruences satisfied by the Bell numbers and polynomials. In particular, we establish the conguences of Touchard, Comtet and Radoux as well as a (new) supercongruence conjectured by M. Zuber.

1 Some polynomial congruences

In this note, p will always denote a fixed prime number and A will either be the ring $\mathbf{Z}$ of integers or the ring $\mathbf{Z}_p$ of p-adic integers. Let $f(x),\, g(x) \in A[x]$ be two polynomials in one variable x and coefficients in the ring A.

Lemma 1.1.- *If* $f(x) \equiv g(x) \;\; mod \;\; p^\nu A[x]$ *for some integer* $\nu \geq 1$*, then*

$$f(x)^p \equiv g(x)^p \;\; mod \;\; p^{\nu+1}A[x].$$

Proof.- By hypothesis

$$f(x) = g(x) + p^\nu h(x) \quad \text{where} \;\; h(x) \in A[x].$$

Hence

$$f(x)^p = (g(x) + p^\nu h(x))^p = g(x)^p + p^{\nu+1} r(x) \quad \text{with} \;\; r(x) \in A[x],$$

and

$$f(x)^p \equiv g(x)^p \;\; mod \;\; p^{\nu+1}A[x].$$

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