A Note on Tensor Products of Polar Spaces Over Finite Fields.

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Abstract

A symplectic or orthogonal space admitting a hyperbolic basis over a finite field is tensored with its Galois conjugates to obtain a symplectic or orthogonal space over a smaller field. A mapping between these spaces is defined which takes absolute points to absolute points. It is shown that caps go to caps. Combined with a result of Dye's one obtains a simple proof of a result due to Blokhuis and Moorehouse that ovoids do not exist on hyperbolic quadrics in dimension ten over a field of characteristic two.

Let $k = GF(q)$, q a prime power, and $K = GF(q^m)$ for some positive integer m. Let $V =< x_1, x_2 > \oplus < x_3, x_4 > \oplus \ldots \oplus < x_{2n-1}, x_{2n} >$ be a vector space over K. Let τ be the automorphism of K given by $\alpha^\tau = \alpha^q$ so that $< \tau > = T = Gal(K/k)$. For each $\sigma \in T$ let V^σ be a vector space with basis $x_1^\sigma, x_2^\sigma, \ldots, x_{2n}^\sigma$. Set $M = V \otimes V^\tau \otimes V^{\tau^2} \otimes \ldots \otimes V^{\tau^{m-1}}$. This is a space of dimension $(2n)^m$ over K. Let $\Im = \{1, 2, \ldots, 2n\}^m$ and for $I = (i_1, i_1, \ldots, i_m) \in \Im$, set $x_I = x_{i_1} \otimes x_{i_2}^\tau \otimes x_{i_3}^{\tau^2} \otimes \ldots \otimes x_{i_m}^{\tau^{m-1}}$. Then $B = \{x_I : I \in \Im\}$, is a basis for M.

We next define a semilinear action of τ on M as follows: For $I = (i_1, i_1, \ldots, i_m) \in \Im$, set $I^\tau = (i_{m-1}, i_0, i_1, \ldots, i_{m-2})$ and then for $a \in K, I \in \{1, 2, \ldots, 2n\}^m$ define $(ax_I)^\tau = a^\tau x_{I^\tau}$ and extend by additivity to all of M. Denote by M^T the set of all vectors of M fixed under this action. This is a vector space over k.

Proposition 1: As a vector space over k, $dim_k M^T = (2n)^m$.

Proof: Let $\Omega_1, \Omega_2, \ldots, \Omega_t$ be the orbits of T in B. Then M^T is the direct sum of the fixed points of τ in $< \Omega_i >_K$ for $i = 1, 2, \ldots, t$. Let $\Omega = \Omega_i$ for some $i, 1 \leq i \leq t$ and let $x = x_I$ be in Ω, assume that $< \tau^l >$ is the stablizer of x_I in T and set

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