When are induction and conduction functors isomorphic ?

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Introduction

Let $R = \bigoplus_{g\in G} R_g$ be a G-graded ring. It is well known (see e.g. [D], [M_1], [N], [NRV], [NV]) that in the study of the connections that may be established between the categories R-gr of graded R-modules and R_1-mod (1 is the unit element of G), an important role is played by the following system of functors :
$(-)_1 : R\text{-gr} \to R_1\text{-mod}$ given by $M \mapsto M_1$, where $M = \bigoplus_{g\in G} M_g$ is a graded left R-module,
the induced functor, Ind : R_1-mod $\to R$-gr, which is defined as follows : if $N \in R_1$-mod, then $\text{Ind}(N) = R\otimes_{R_1} N$ which has the G-grading given by $(R\otimes_{R_1} N)_g = R_g \otimes_{R_1} N, \forall g \in G$,
and the coinduced functor, Coind : R_1-mod $\to R$-gr, which is defined in the following way : if $N \in R_1$-mod, then $\text{Coind}(N) = \bigoplus_{g\in G}\text{Coind}(N)_g$, where

$$\text{Coind}(N)_g = \{f \in \text{Hom}_{R_1}({}_{R_1}R_R, N) \mid f(R_h) = 0,\ \forall\ h \neq g^{-1}\}\ .$$

(Note that if G is finite, then $\text{Coind}(N) = \text{Hom}_{R_1}({}_{R_1}R_R, N)$).

It was shown in [N] that the functor Ind is a left adjoint of the functor $(-)_1$ and the unity of the adjunction $\sigma : \mathbf{1}_{R_1\text{-mod}} \to (-)_1 \circ \text{Ind}$ is a functorial isomorphism, and that Coind is a right adjoint of the functor $(-)_1$ and the counity of this adjunction $\tau : (-)_1 \circ \text{Coind} \to \mathbf{1}_{R_1\text{-mod}}$ is a functorial isomorphism.
If the ring R is a G-strongly graded ring (i.e. $R_gR_h = R_{gh}\ \ \forall g, h \in G$) then the functors Ind and Coind are isomorphic. Thus the following question naturally

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