Substitutions, Partial Isometries of $\mathbb{R}$, and Actions on Trees

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Abstract

In this paper we outline a connection between substitution dynamical systems and systems of partial isometries of $\mathbb{R}$. Let τ be a primitive substitution and (X, T) the associated minimal subshift. Associated to each real geometric realization $\mathcal{G}$ of τ is a system of partial isometries $\mathcal{I}_{\mathcal{G}}$. The subshift (X, T) generated by τ provides a symbolic coding of the dynamics of $\mathcal{I}_{\mathcal{G}}$. This allows us to deduce various dynamical properties of the system: Minimality, classification (interval exchange, homogeneous, or exotic), orbit structure, independence of generators, and self similarity. We show by example that each type of minimal system of partial isometries can arise from geometric realizations of substitutions on three letters. These systems give rise to interesting (non-simplicial) geometric actions of free groups on real trees.

1 Introduction

In [16] we exhibit a connection between substitution subshifts and isometric actions of free groups on $\mathbb{R}$-trees. Our construction begins with a minimal interval translation mapping $f : [0, 1] \to [0, 1]$ on three intervals. This transformation was originally studied by M. Boshernitzan and I. Kornfeld in [4]. As noted in [4], the dynamics of f is symbolically coded in terms of the substitution $\tau(1) = 2$, $\tau(2) = 3111$, $\tau(3) = 311$. The mapping f is "self-similar" in the sense that there exists a proper subinterval I_0 of $I = [0, 1]$ for which the first return map f_0 is an interval translation mapping

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