Transitivity, dense orbit and discontinuous functions*

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The main "ingredient" in Devaney's definition of chaos is transitivity (see [3]). Banks, Brooks, Cairns, Davis and Stacey [1] demonstrated the redundancy of sensitive dependence on initial conditions (the most popularly understood hypothesis of Devaney). They showed as a consequence that chaos is a topological property (not metric, as one could think about the original definition). Moreover, transitivity implies chaos for continuous functions on intervals (we refer to [4] for a simple proof). We recall that a map $f : M \to M$ is *transitive* if for any pair of non-empty open sets U and V in M, there is some $k > 0$ with $f^k(U) \cap V \neq \emptyset$. Here M denotes a metric space and f^k is f composed with itself k times. Transitivity can be seen, in words of Banks et al., as an irreducibility condition. It is worth mentioning that other alternatives to transitivity had been provided by A. Crannell [2], which can be regarded as more intuitive properties.

For Baire separable metric spaces M (which is usually the case) and continuous f, transitivity implies the existence of a point $x \in M$ whose orbit is dense in M. Let us notice that separability of M is obviously necessary to get the existence of a dense orbit. The phenomenon of dense orbit is much more intuitive than transitivity. On the other hand there are discontinuous maps which are interesting from the point of view of discrete dynamical systems (e.g., Baker's function $B(x) := 2x$ for $0 \leq x \leq 1/2$, $B(x) := 2x-1$ for $1/2 < x \leq 1$). But Baire's Theorem is not applicable in general for discontinuous functions. This leads us to the following question: For which type of discontinuous functions are transitivity and existence of dense orbits equivalent?

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