Caps embedded in the Klein quadric

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1 Introduction

Let $PG(N,q)$ be the projective space of dimension N over the finite field $GF(q)$. A *k–cap* K in $PG(N,q)$ is a set of k points, no three of which are collinear [14], and a k–cap is called *complete* if it is maximal with respect to set–theoretic inclusion. The maximum value of k for which there exists a k–cap in $PG(N,q)$ is denoted by $m_2(N,q)$ [14]. This number $m_2(N,q)$ is only known, for arbitrary q, when $N \in \{2,3\}$. Namely, $m_2(2,q) = q+1$ if q is odd, $m_2(2,q) = q+2$ if q is even, and $m_2(3,q) = q^2+1$, $q > 2$. With respect to the other values of $m_2(N,q)$, apart from $m_2(N,2) = 2^N$, $m_2(4,3) = 20$, $m_2(5,3) = 56$ and $m_2(4,4) = 41$ [2], only upper bounds are known. Finding the exact value for $m_2(N,q)$, $N \geq 4$ and constructing an $m_2(N,q)$–cap seems to be a very hard problem. In the last few years there has been a certain interest in caps embedded in the Klein quadric $\mathcal{K}$ of $PG(5,q)$ considered as ambient space, and the main purpose is to find lower and upper bounds for a complete cap embedded in $\mathcal{K}$. In this direction, Blokhuis and Sziklai [3] proved a lower bound for the smallest complete cap of the Klein quadric. Precisely such a cap has size at least $const{\cdot}q^{12/7}$. In 1997, Cossidente, Hirschfeld and Storme [8] constructed a cap of size $2q^2+q+1$ of $\mathcal{K}$ obtained by gluing together two suitable Veronese surfaces. If we assume q even, it is always possible to extend such a cap to a complete $2(q^2+q+1)$–cap of $\mathcal{K}$ [5]. This seems to be the unique known example of smallest complete cap of $\mathcal{K}$. On the other hand Glynn [12] proved (using the Klein correspondence between lines of $PG(3,q)$ and points of $PG(5,q)$) that any line orbit of a Singer cyclic group of $PG(3,q)$ corresponds to a cap of size

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