THE FUNDAMENTAL THEOREM OF ALGEBRA – PROOFS VIA "THE CREEPING LEMMA"

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Let $\mathbb{C}$ be the complex plane. A complex polynomial function is a function $P\colon \mathbb{C} \to \mathbb{C}$ defined by

$$P(z) = \sum_{k=0}^{n} a_{n-k} z^{n-k},$$

where $a_0, \cdots, a_n \in \mathbb{C}$. The Fundamental Theorem of Algebra states that a nonconstant complex polynomial function has a zero. Over the years, a number of proofs of this theorem have been discovered [1]. In this note, the so-called "Creeping Lemma" [3] is used to present two elementary proofs of the Fundamental Theorem. The proofs can be presented in a first analysis course (or advanced calculus course) as a nice application of diffeomorphism and the concept of least upper bound.

It is known that a complex polynomial function P is a closed function, i.e. that $P(A)$ is a closed subset of $\mathbb{C}$ when A is a closed subset of $\mathbb{C}$. The proof for non-constant P combines the Bolzano-Weierstrass property of $\mathbb{C}$ with the continuity of P and the observation that $|P(z)|$ gets arbitrarily large as $|z|$ gets arbitrarily large [2].

Here the "Creeping Lemma" is applied

(i) to provide a direct proof of The Fundamental Theorem of Algebra, and
(ii) to prove that a non-constant complex polynomial function is an open function, so $P(\mathbb{C}) = \mathbb{C}$, since $P(\mathbb{C})$ is a nonempty, closed, and open subset of the connected space $\mathbb{C}$ (another proof of the Fundamental Theorem of Algebra).

In the sequel, the set of critical points of a complex polynomial P, $\{z \in \mathbb{C} : P'(z) = 0\}$, will be denoted by M.

Theorem 1. *The Fundamental Theorem of Algebra*: A non-constant complex polynomial function has a zero.

Proof. The set of critical points, M and its image $P(M)$ are both finite. Let $p = P(a)$ with $a \notin M$. Let $\sigma\colon [0,1] \to \mathbb{C}$ be a curve such that $\sigma(0) = p$, $\sigma(1) = O$, and $\sigma([0,1)) \subset \mathbb{C} - P(M)$, an open connected set. It will be enough to show that there is a curve $\tilde{\sigma}\colon [0,1] \to \mathbb{C}$ with $\tilde{\sigma}(0) = a$ and $P \circ \tilde{\sigma} = \sigma$.

Choose a neighborhood U of a such that $P\colon U \to P(U)$ is a diffeomorphism with inverse g. Then, $\tilde{\sigma} = g \circ \sigma$ is defined