SOLUTIONS

No problem is ever permanently closed. Any comments, new solutions, or new insights on old problems are always welcomed by the problem editor.

149. [2004, 129] *Proposed by Joe Howard, Portales, New Mexico and Les Reid, Southwest Missouri State University, Springfield, Missouri.*

Let A, B, C be the angles of a triangle. Show

$$\begin{aligned}3+\cos A+\cos B+\cos C &\ge 2(\sin A\sin B+\sin B\sin C+\sin C\sin A)\\ &\ge 9(\cos A+\cos B+\cos C-1)\end{aligned}$$

with equality if and only if the triangle is equilateral.

Solution by the proposers. Let r, R, and s denote the inradius, circumradius, and semiperimeter, respectively. Euler's inequality ($R \ge 2r$) is well known. The following can be found in *Crux Mathematicorum,* [2029] 22 (3) (1996), p. 130 and on pages 45 and 55–56 of D. S. Mitrinović, J. E. Pečurić, and V. Volenec, *Recent Advances in Geometric Inequalities,* Kluwer Academic Pub., 1989.

$$\sin A\sin B+\sin B\sin C+\sin C\sin A=\frac{s^2+4Rr+r^2}{4R^2} \tag{1}$$

$$\cos A+\cos B+\cos C=1+\frac{r}{R} \tag{2}$$

$$4R^2+4Rr+3r^2\ge s^2\ge 16Rr-5r^2 \tag{3}$$

Using (1) and (2), the given inequality is equivalent to

$$4+\frac{r}{R}\ge 2\left(\frac{s^2+4Rr+r^2}{4R^2}\right)\ge 9\left(\frac{r}{R}\right)$$

which is equivalent to

$$8R^2-2Rr-r^2\ge s^2\ge 14Rr-r^2.$$