AN INFORMAL APPROACH TO FORMAL INNER PRODUCTS

Richard Hammack

The fact that linear algebra is a distillation of our perception of physical space gives teachers and students of the subject a vast source of heuristic arguments. Almost every concept or result in finite-dimensional linear algebra can be illustrated or explained with a well-chosen geometric figure or schematic diagram.

The formal definition of an inner product seems to be an exception. Students who appreciate geometric reasoning are often mystified when, usually in their second linear algebra course, they first encounter this definition. They agree that an inner product gives a space a sense of length and orthogonality, but they are perplexed by the fact that its formal properties seem to have no geometric rationale. "Why," they ask, "is the definition phrased the way it is?"

In teaching a second-semester course recently, I devised an answer to that question. My strategy is the reverse of the standard one. I begin with intuitive notions about length and orthogonality, and derive the formal inner product from them, rather than the other way around. This article is an outline of my approach.

When introducing the formal definition of an inner product, I first remind the students of the standard inner product on $\mathbb{R}^n$. They have been exposed to it in their previous courses, and have an understanding of how it gives $\mathbb{R}^n$ metric properties. Then I give the formal definition of an inner product, pointing out that the standard inner product on $\mathbb{R}^n$ satisfies this definition.

An *inner product* on a vector space V over a field $\mathbb{F}$, where $\mathbb{F}$ is either the real numbers $\mathbb{R}$ or the complex numbers $\mathbb{C}$, is a function $\langle\ ,\ \rangle\colon V \times V \to \mathbb{F}$ with the following properties. If $u, v, w \in V$, and $c \in \mathbb{F}$, then

1. $\langle u+v, w\rangle = \langle u, w\rangle + \langle v, w\rangle$
2. $\langle cv, w\rangle = c\langle v, w\rangle$
3. $\langle w, w\rangle > 0$ if $w \neq 0$
4. $\langle v, w\rangle = \overline{\langle w, v\rangle}$.

Further, from properties 1, 2, and 4, it follows that

5. $\langle u, v+w\rangle = \langle u, v\rangle + \langle u, w\rangle$
6. $\langle v, cw\rangle = \overline{c}\langle v, w\rangle$.

The bar represents complex conjugation, which is superfluous when $\mathbb{F} = \mathbb{R}$ [1].