SOLUTIONS

No problem is ever permanently closed. Any comments, new solutions, or new insights on old problems are always welcomed by the problem editor.

137. [2002, 210] *Proposed by José Luis Díaz, Universidad Politécnica de Cataluña, Barcelona, Spain.*

Find all non-negative integers a, b, and c such that $a+b+c$ and abc are consecutive integers.

Solution by Russell Euler and Jawad Sadek, Northwest Missouri State University, Maryville, Missouri. Assume without loss of generality that $a \leq b \leq c$. If $2 \leq a \leq b \leq c$, then writing $b = a+m$, $c = a+n$ and $abc = a+b+c+1$ implies

$$a^3 + (m+n)a^2 + amn = 3a + m + n + 1.$$

This is a contradiction as the left-hand side is larger than the right-hand side when $a \geq 2$ and m and n are positive integers. We get a similar contradiction when we write $abc = a+b+c-1$. It follows that a has to be either 0 or 1.

Case I. If $a = 0$, then $abc = a+b+c+1$ gives a contradiction. But $abc+1 = a+b+c$ implies $1 = b+c$ which implies $b = 0$ and $c = 1$. So the solution in this case is $a = b = 0$ and $c = 1$.

Case II. If $a = 1$, then either $bc = b+c+2$ or $bc = b+c$.

(i) If $bc = b+c$, then $b = b/c + 1$ and so we must have $b = c$. This implies that $b = 2$ and $c = 2$.

(ii) If $bc = b+c+2$, then $b = (b+2)/c + 1$. This implies that $b+2 = c$. By substitution, $b(b+2) = 2b+4$ or $b^2 = 4$ and so $b = 2$ and $c = 4$.

The solutions are $(0,0,1)$, $(1,2,2)$, and $(1,2,4)$.

Also solved by J. D. Chow, Edinburg, Texas; Joe Howard, Portales, New Mexico; James T. Bruening, Southeast Missouri State University, Cape Girardeau, Missouri; and the proposer.