UNIVERSAL FORMS: THE FOUR-SQUARE THEOREM AND ITS GENERALIZATIONS

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1. Introduction. The problem of writing a positive integer as a sum of squares has been a source of fascination for centuries, attracting the attention of some of the finest mathematical minds throughout history. One well-known result along these lines is the "Four-Square Theorem."

Theorem 1. If n is a positive integer, then there exist integers x, y, z, and w such that $n = x^2 + y^2 + z^2 + w^2$.

As many readers are no doubt aware, the first correct proof of the theorem is due to Lagrange. However, the history of this theorem, and its impact on the development of modern number theory, are less well-known. In Section 1 of this paper we give a historical account of the Four-Square Theorem and present a proof due to Euler. In Section 2, we present a generalization due to Ramanujan. For references we list the most readily accessible, most of which can be found in any college library. For biographical information on the mathematicians involved, we recommend [13 or 14]. Finally, although we attempt to follow the thread of reasoning of the original work, where convenient we use modern terminology.

2. The Four-Square Theorem. The first explicit statement of the Four-Square Theorem was given in 1621 by Claude Bachet, who in that year published a Latin translation of Diophantus' *Arithmetic*. In his commentary, Bachet stated that he had verified the theorem for all numbers up to 325, and remarked that Diophantus himself seemed to have known the theorem. The reason for doing so was that, although Diophantus gives necessary conditions for an integer to be the sum of two squares, or of three squares, no precondition is given for an integer to be the sum of four squares. Among the mathematicians to read Bachet's text was Fermat, and it was in the margins of this book that he wrote many of his most famous results and conjectures (including his fabled "last theorem"). In the 1630's, Fermat began a long and fruitful correspondence with Mersenne in which he formulated many results and conjectures concerning the represenation of integers by quadratic forms [13]. In a letter from 1636 [7], Fermat stated that he had a proof of the theorem, presumably by his method of descent. Like Bachet, Fermat also credited knowledge of the theorem to Diophantus.