MEANS AND THEIR ENDS

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The Arithmetic-Geometric Mean Inequality [1] guarantees that

$$GM(a_1, a_2, \ldots, a_n)/AM(a_1, a_2, \ldots, a_n) \leq 1$$

for any finite set of positive terms $\{a_1, a_2, \ldots, a_n\}$. (Here GM and AM denote the geometric and arithmetic means, respectively.) Except in the case where the numerator and denominator are equal, however, we are given no clue as to what the value of that ratio might be. In [2] I showed that for any positive real number s

$$\lim_{n\to\infty} \frac{GM(1^s, 2^s, \ldots, n^s)}{AM(1^s, 2^s, \ldots, n^s)} = \frac{s+1}{e^s}.$$

The special case $s = 1$ leads to the well-known result that

$$\lim_{n\to\infty} \frac{\sqrt[n]{n!}}{n} = \frac{1}{e}.$$

In fact, a more general result holds for polynomial sequences and also for positive power sequences.

Proposition 1. If $a_k = c_s k^s + c_{s-1}k^{s-1} + \cdots + c_1 k + c_0$ is positive for each k or if $a_k = ck^s$ for c and s positive real numbers, then

$$\lim_{n\to\infty} \frac{GM(a_1, a_2, \ldots, a_n)}{AM(a_1, a_2, \ldots, a_n)} = \frac{s+1}{e^s}. \tag{1}$$

Furthermore, if we define arithmetic and geometric means for a continuous positive function f by

$$AM(f, n) = \frac{1}{n}\int_0^n f(x)\,\mathrm{d}x \quad \text{and} \quad GM(f, n) = \exp\left[\frac{1}{n}\int_0^n \ln f(x)\,\mathrm{d}x\right]$$