ANOTHER PROOF OF THE CHANGE OF VARIABLE FORMULA FOR d-DIMENSIONAL INTEGRALS

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The Volume 105, Number 7 issue of *The American Mathematical Monthly* published a new proof of the change of the variable formula for d-dimensional integrals

$$\int_{\mathbb{R}^d} f(x)\, d\lambda(x) = |\det(A)| \int_{\mathbb{R}^d} f(Ax)\, d\lambda(x) \tag{1}$$

with an invertible matrix A, based on group theoretical arguments [1]. In this note we provide another proof of (1) to illustrate an application of elementary measure theory and the singular value decomposition.

We use the same notation as in [1]. The measure $\lambda \circ A^{-1}$, which is defined by $(\lambda \circ A^{-1})(B) = \lambda(A^{-1}(B))$ for all Borel sets B, is equivalent to (i.e., absolutely continuous with respect to each other) the Lebesgue measure λ for any $A \in GL(d, \mathbb{R})$.

Proposition 1.

$$\int_{\mathbb{R}^d} f(Ax)\, d\lambda(x) = \int_{\mathbb{R}^d} f(x)\, d(\lambda \circ A^{-1})(x). \tag{2}$$

Proof. Let $f = 1_B$, where 1_B is the characteristic function of B. Then

$$\int_{\mathbb{R}^d} 1_B(Ax)\, d\lambda(x) = \lambda(A^{-1}(B)) = \int_{\mathbb{R}^d} 1_B(x)\, d(\lambda \circ A^{-1})(x),$$

i.e., (2) is true for all characteristic functions, which implies that (2) is satisfied by all simple functions. Since f is the limit of a sequence of simple functions [3], using a limiting process we see that (2) is valid for all integrable functions f.

Proposition 2. If $A \in GL(d, \mathbb{R})$ is orthogonal, then $(\lambda \circ A^{-1})(B) = \lambda(B)$ for all Borel sets B.

Proof. Every orthogonal matrix is a product of several rotations and reflections which do not change the Lebesgue measure of a Borel set.