MINIMAL SURFACES: A DERIVATION OF THE MINIMAL SURFACE EQUATION FOR AN ARBITRARY C^2 COORDINATE CHART

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1. Introduction. The study of minimal surfaces is an exciting and active area of mathematical research. Much of the excitement comes from the desire to understand the geometry of soap films, which naturally assume the shapes of minimal surfaces. In this paper, we recall Lagrange's minimal surface equation and derive a more general minimal surface equation. We begin with two important theorems that are well known in the study of minimal surfaces as found in [1].

Theorem 1. (Lagrange, 1760). Let f be a C^2, real-valued function on a planar domain R, and let $\phi: R \to \mathbb{R}^3$ be defined by $\phi(x,y) = (x, y, f(x,y))$. Suppose that the graph of f, i.e., the image $\phi(R)$, is area-minimizing. Then f satisfies the following minimal surface equation.

$$f_{xx}(1+f_y^2) - 2f_x f_y f_{xy} + f_{yy}(1+f_x^2) = 0.$$

A partial converse of this theorem was later proved for specific domains and can be stated as follows.

Theorem 2. (Federer, 1969). Using the notation in Theorem 1, if f satisfies the graph minimal surface equation on a convex domain R, then its graph $\phi(R)$ is area-minimizing.

Each of these theorems is vital to the study of minimal surfaces. However, because of the severe restrictions on the types of coordinate charts, an additional minimal surface equation is necessary for further study. The more general minimal surface equation that would be satisfied by any arbitrary area-minimizing C^2 coordinate chart $\phi: R \to \mathbb{R}^3$ has over 600 terms. (We verified this using Maple©.) Thus, it has never been written down in an easily accessible form. Using the calculus of variations for vector-valued functions, it is this seemingly abominable general minimal surface equation that we derive in this paper and express fairly simply. In addition, we will use this equation to verify that certain surfaces described by general coordinate charts are minimal.