SOLUTIONS

No problem is ever permanently closed. Any comments, new solutions, or new insights on old problems are always welcomed by the problem editor.

40. [1991, 150; 1992, 150–151] *Proposed by Stan Wagon, Macalester College, St. Paul, Minnesota.*

A tetrahedron is a geometric solid with 4 vertices, 6 edges, and 4 triangular faces. A Heron triangle is one whose sides and area are integers. A Heron tetrahedron is one having Heron triangles as faces and whose volume is an integer.

(a) Show that if $\triangle ABC$ is acute, then a tetrahedron exists with each of its faces congruent to $\triangle ABC$.

(b)* John Leech has shown that a Heron tetrahedron exists: Let $\triangle ABC$ have sides 148, 195, and 203 and let T be the tetrahedron obtained from this triangle as in (a). Then each face of T has integer area and T has integer volume. The following question is inspired by Jim Buddenhagen's investigation of Heron triangles whose area is a square. Question: Is there a Heron tetrahedron whose volume is a perfect square or perfect cube?

Comment by Les Reid, Southwest Missouri State University, Springfield, Missouri. Once the existence of a Heron tetrahedron is known, it's relatively easy to construct a Heron tetrahedron whose volume is a perfect square (or, in fact, any perfect power whose exponent is not a multiple of 3). In general, if we scale the tetrahedron by a factor of L, the volume will increase by a factor of L^3 and the area by a factor of L^2 (so it will still be an integer). If we choose L to be the square-free part of the volume, the volume will be a perfect square. For example, starting with Leech's tetrahedron having four congruent faces with edges of length 148, 195, and 203, it's volume is

$$611520 = 2^6 * 3 * 5 * 7^2 * 13,$$

whose square-free part is $3 * 5 * 13$. Therefore, the corresponding tetrahedron with edges of length 28860, 38025, and 39585 will have a volume of

$$2129400^2.$$

A similar argument works as long as the exponent of the power is not a multiple of 3. For example, if we want the volume to be a fifth power (and begin with Leech's tetrahedron), we would choose

$$L = 2^x * 3^y * 5^z * 7^s * 13^t,$$