INTEGRATION OF RATIONAL FUNCTIONS BY THE SUBSTITUTION $\mathbf{x = u^{-1}}$

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Integration of a proper rational function by the method of partial fractions usually involves tedious calculations when its denominator contains repeated linear or quadratic factors. Often this difficulty can be alleviated by the substitution $x = u^{-1}$ which is illustrated in the following examples.

Example 1. Integrating the function $R(x) = (x^4 - x^3)^{-1}$ by the substitution $x = u^{-1}$ changes it to $R_1(u) = u^4/(1-u)$ and shows that

$$\int \frac{dx}{x^4 - x^3} = \int \frac{u^4}{1-u} \cdot \frac{du}{-u^2} = \int \frac{u^2}{u-1} du$$

$$= \int \left(u + 1 + \frac{1}{u-1} \right) du = \frac{u^2}{2} + u + \ln|u-1| + C.$$

The answer is

$$\int \frac{dx}{x^4 - x^3} = \frac{1}{2x^2} + \frac{1}{x} + \ln\left|\frac{x-1}{x}\right| + C.$$

Example 2. The denominator of the function $R(x) = (3x-1)/x^4(x-1)^2$ contains two repeated linear factors, and in this case the substitution $x = u^{-1}$ especially enhances integration giving

$$\int \frac{(3x-1)dx}{x^4(x-1)^2} = \int \frac{(3u^3 - u^4)u^2}{(u-1)^2} \cdot \frac{du}{-u^2}$$

$$= \int \frac{u^3(u-3)}{(u-1)^2} du.$$