# GRAPHS AND MATRICES IN THE STUDY OF FINITE (TOPOLOGICAL) SPACES

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**Introduction.** In a first course in topology (e.g. [10]), one invariably comes across a finite topology, i.e. a topology on a space $X$ with $n < \infty$ points. Beginning students (and many instructors) are quite surprised when first told that, for instance, if $X$ has just 6 points, there are 209,527 possible topologies on $X$. This paper is written in part for those who wonder "why so big a number?", and "how is it obtained?" In fact, there are 115,617,051,977,054,267,807,460 topologies possible on a set with $n = 14$ elements, and this seems to be the largest $n$ for which the number of topologies is known (see [4]).

The main purpose of this article, however, is not to study this specific enumeration problem. We instead focus on a productive relationship between graph theory, matrix algebra, and finite topologies. While teaching an introductory topology class, we chanced on [13], which alluded to a graph theoretic approach to the study of finite topologies: each topology on $X$ can be identified with a certain directed graph with $n$ nodes (see also [3]). This gives a nice way to literally visualize a topology. We then show how the adjacency matrix associated to the graph (defined in section 3) provides many surprises. For one, the left "eigenvectors" of the matrix directly correspond to the open sets in the topology; and the right "eigenvectors" correspond to the closed sets. In addition, matrices will also allow us to put a natural topology on the space of finite topologies. For much of the discussion, the figures play a crucial role in understanding. (Many of our discoveries have since turned out to be previously known in the fairly scattered literature on this subject. Our approach, however, with its emphasis on the graph and its adjacency matrix, differs from those taken in most of the literature. Our approach allows for some one-line proofs of published results. Many of our observations concerning adjacency matrices, such as those dealing with fineness of topologies and product topologies, seem to be new.) A knowledge of the basic definitions in undergraduate topology, and some discrete mathematics, is all that will be assumed.

In the final two sections, we discuss an (apparently new) enumeration problem associated to finite topologies, and include some preliminary results; then propose a conjecture; and finally raise several questions about finite topological spaces.

**1. Finite Topologies, Pre- and Partial Orders, and Directed Graphs.** We begin with the observation, slightly generalizing [1] and used in [3], that for a