THE ARITHMETIC-MEAN – GEOMETRIC-MEAN INEQUALITY DERIVED FROM CLOSED POLYNOMIAL FUNCTIONS

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An activity which is encouraged in the teaching and study of mathematics is that of exploring how classical results may be derived from other concepts. The Arithmetic-Mean – Geometric-Mean Inequality (AMGM) states

$$\prod_{m=1}^{n} x_m \leq \left(\frac{\sum_{m=1}^{n} x_m}{n} \right)^n$$

for all positive integers n and nonnegative reals $x_1, \ldots, x_n$, with equality if and only if $x_k = x_j$ for all k and j, where $\prod_{m=1}^{n} x_m$ and $\sum_{m=1}^{n} x_m$ denote the product and sum of the numbers $x_1, \ldots, x_n$, respectively. The purpose of this note is to show that this classical inequality is an easy consequence of the fact that the function P defined on $\mathbb{R}^n$, Euclidean n-space, by $P(x_1, \ldots, x_n) = \sum_{m=1}^{n} |x_m|$ is a closed function, i.e. if A is a closed subset of $\mathbb{R}^n$, then $P(A) = \{P(x_1, \ldots, x_n) : (x_1, \ldots, x_n) \in A\}$, the image of A under P, is a closed subset of $\mathbb{R}$. When restricted to $\mathbb{R}^n_+ = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_m \geq 0 \text{ for each } m\}$, P is a polynomial function. We found this proof while studying closed functions between Euclidean spaces. Although we do not know if the proof is new, it does represent an excellent opportunity for students to see continuous functions, closed functions, and greatest lower bound working together. We also give a proof using compactness and continuity of the function Q defined on $\mathbb{R}^n$ by $Q(x_1, \ldots, x_n) = \prod_{m=1}^{n} x_m$ (see [1, 2]).

The following result will be applied to establish the AMGM.

Lemma. The function P defined on $\mathbb{R}^n$ by $P(x_1, \ldots, x_n) = \sum_{m=1}^{n} |x_m|$ is a closed function.

Proof. Let $A \subset \mathbb{R}^n$ be closed, let $r \in \mathbb{R}$, and let $\{(x_1^k, \ldots, x_n^k)\}_{k=1}^{\infty}$ be a sequence in A satisfying $P(x_1^k, \ldots, x_n^k) \to r$. Then $\{P(x_1^k, \ldots, x_n^k) : k = 1, \ldots\}$ is bounded and hence, the sequence $\{(x_1^k, \ldots, x_n^k)\}_{k=1}^{\infty}$ is a bounded sequence in A. By the Bolzano-Weierstrass Theorem, this sequence has a subsequence, which we again call $\{(x_1^k, \ldots, x_n^k)\}_{k=1}^{\infty}$, such that $(x_1^k, \ldots, x_n^k) \to (y_1, \ldots, y_n)$ and $(y_1, \ldots, y_n) \in A$,