LINEARIZATION OF TRIGONOMETRIC POLYNOMIALS AND INTEGRALS

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1. The Method. One of the problems encountered in calculus courses is the computation of simple integrals. The existence of a primitive for a continuous function on an interval is a well-known theorem (see [3]), but the actual computation is sometimes quite hard to perform, even impossible. In [2] a geometric approach for computing one particular trigonometric integral is presented and an interesting method for computing trigonometric integrals is given in [1]. We present a way to compute integrals of trigonometric polynomials, via a linearization of these polynomials using complex numbers and Euler's formula (the method is simple, but we did not find any exposition of it in classical textbooks).

Recall that for $x \in \mathbb{R}$, $e^{ix} = \cos x + i \sin x$. By DeMoivre's formula, we have for any $x \in \mathbb{R}$, $e^{-ix} = \cos x - i \sin x$ and for any $n \in \mathbb{Z}$, $(e^{ix})^n = e^{nix}$. Therefore,

$$\begin{cases} \cos x = \frac{1}{2}(e^{ix} + e^{-ix}) \\ \sin x = \frac{1}{2i}(e^{ix} - e^{-ix}). \end{cases}$$

These are the so-called Euler's formula.

By Newton's binomial development, we can now compute any positive integral power of $\cos x$ and of $\sin x$ and any product of such powers as a linear combination of powers of e^{ix} and e^{-ix}.

1. Let $f(x) = \cos^p x$. Then

$$\begin{aligned} \cos^p x &= \left[\frac{1}{2}(e^{ix} + e^{-ix})\right]^p \\ &= \frac{1}{2^p} \sum_{k=0}^{p} \binom{p}{k} (e^{kix} \cdot e^{-(p-k)ix}) \\ &= \frac{1}{2^p} \sum_{k=0}^{p} \binom{p}{k} e^{(2k-p)ix} \\ &= \frac{1}{2^p} \sum_{k=0}^{p} \binom{p}{p-k} e^{(2k-p)ix}. \end{aligned}$$