SOLUTIONS

No problem is ever permanently closed. Any comments, new solutions, or new insights on old problems are always welcomed by the problem editor.

113*. [1998, 46] *Proposed by Kamal Jain, Georgia Institute of Technology, Atlanta, Georgia.*

Find all ordered pairs (a, b) such that

$$\tan(a\pi) = b$$

and a and b are rational numbers.

Solution by Bob Prielipp, University of Wisconsin - Oshkosh, Oshkosh, Wisconsin.

In his article "Rational Values of Trigonometric Functions" [see pp. 507–508 of *The American Mathematical Monthly*, 52 (1945)], J. M. H. Olmsted proved that the only rational values of $\tan(a\pi)$ (where a is a rational number) are 0 and ± 1.

Thus, if a and b are rational numbers then (a, b) is a solution of $\tan(a\pi) = b$ if and only if (a is an arbitrary integer and $b = 0$) or ($a = \frac{1}{4} + k$ where k is an arbitrary integer and $b = 1$) or ($a = -\frac{1}{4} + k$ where k is an arbitrary integer and $b = -1$).

Also, on the pages leading up to p. 41 of his book *Irrational Numbers*, (Carus Monograph #11) The Mathematical Association of America (distributed by John Wiley and Sons, Inc.), 1963, Ivan Niven proved that if θ is rational in degrees, say $\theta = 2\pi r$ for some rational number r, then the only rational values of the trigonometric functions of θ are as follows: $\sin\theta$, $\cos\theta = 0, \pm\frac{1}{2}, \pm 1$; $\sec\theta$, $\csc\theta = \pm 1, \pm 2$; $\tan\theta$, $\cot\theta = 0, \pm 1$.

114. [1998, 46] *Proposed by Kenneth B. Davenport, 301 Morea Road, Frackville, Pennsylvania.*

(a) Prove that

$$\int_0^\infty \frac{1}{1+x^2} \cdot \frac{4}{4+x^2} \cdot \dots \cdot \frac{n^2}{n^2+x^2} dx = \frac{\pi}{2} \frac{n}{2n-1}.$$