FINDING PYTHAGOREAN TRIPLE PRESERVING MATRICES

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1. Introduction. When we multiply a Pythagorean triple with a 3×3 matrix we obtain another triple, but will it be Pythagorean? A problem posed in 1987 showed an example of a 3×3 matrix

$$A = \begin{pmatrix} 2 & 1 & 2 \\ 1 & 2 & 2 \\ 2 & 2 & 3 \end{pmatrix}$$

which converts a Pythagorean triple into a Pythagorean triple [1]. For example $(3,4,5)A = (20,21,29)$, which is again a Pythagorean triple. Indeed one can verify that if $(a,b,c)A = (d,e,f)$ and $a^2+b^2=c^2$, then $d^2+e^2=f^2$. In other words the matrix A "preserves" Pythagorean triples.

In this paper we will find matrices which "preserve" Pythagorean triples. To be specific, we will find necessary and sufficient conditions that a 3×3 matrix preserves Pythagorean triples. In the second paper we will discuss construction of matrices which play a prescribed role, i.e. given two Pythagorean triples, say X and Y, we construct a matrix A such that $XA = Y$.

2. Preliminary Definitions. We define a Pythagorean Triple (PT) as a triple (a,b,c) where a, b, and c are positive integers and $c^2 = a^2+b^2$. If in addition, a, b, and c have no factor in common, the triple is called a Primitive Pythagorean Triple (PPT). By our definition both $(3,4,5)$ and $(4,3,5)$ are PPTs. To keep our analysis simple, it is necessary to distinguish between these two types. We will say $(3,4,5)$ is of type A and $(4,3,5)$ is of type B, i.e., a PPT (a,b,c) is of type A or type B according as a or b is an odd integer. Furthermore, we will denote them by PPTA and PPTB, respectively. A matrix that converts a PPT (of type A or B) into a PPT (of type A or B) will be called a Pythagorean Triple Preserving Matrix and will be denoted by PTPM. We note that the matrix A shown above converts a PPTA into a PPTB. The object of this paper is to find all PTPMs.