A COMPOSITION PROBLEM INVOLVING ANALYTIC FUNCTIONS

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1. Introduction. One of the beautiful principles of Fourier analysis maintains that when an analytic function ϕ acts upon a function f continuous on the unit circle T in the complex plane, the composition $\phi(f)$ frequently inherits "nice" properties possessed by f. An example of this idea is the Wiener-Levy Theorem, which asserts that if f has an absolutely convergent Fourier series and if ϕ is a function analytic in a neighborhood of $f(T)$, then the composition $\phi(f)$ has an absolutely convergent Fourier series as well [4].

In this note we consider a composition problem in which the "nice" property can be expressed in terms of analyticity or vanishing Fourier coefficients. Specifically, we let $A(D)$ (sometimes referred to as the "disk algebra") denote the algebra of functions that are continuous on T and that possess analytic extensions into the open unit disk D. Equivalently, owing to the Poisson integral, a function f belongs to $A(D)$ if and only if it is continuous on T and has the property that its Fourier coefficients

$$\hat{f}(n) = \frac{1}{2\pi}\int_0^{2\pi} f(t)e^{-int}dt$$

are equal to zero for all $n < 0$.

Clearly if f belongs to $A(D)$ and if φ is analytic in a neighborhood of $f(\overline{D})$, then $g = \varphi(f)$ belongs to $A(D)$, and one may express its extension in D as $g(z) = \varphi(f(z))$. Moreover, the elementary examples consisting of the pairs $\varphi(z) = z^2$, $f(z) = \sqrt{z}$, and $\varphi(z) = e^z$, $f(z) = \log(z)$ (where any branch of f is chosen) illustrate how an analytic function φ can map a function f not in $A(D)$ to a function $g = \varphi(f)$ back in this class.

Yet for neither of these two examples is the function f continuous on the unit circle. A more difficult question, one raised by Forelli [2], arises when one asks whether, for such an arbitrary function φ, there exists a function f continuous on T and not in $A(D)$ that is mapped to a function $g = \varphi(f)$ in $A(D)$. In terms of analyticity, $g = \varphi(f)$ has an analytic extension $g(z)$ in D even though f, while continuous on T, does not possess such an extension. In terms of Fourier coefficients,