A PERPLEXING FINITE CONTINUED FRACTION

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Let

$$H_n = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

be the nth Harmonic number. Although there is no closed form expression for H_n in terms of common mathematical functions, it is known that H_n *can* be expressed in terms of Stirling numbers. Specifically,

$$H_n = \frac{1}{n!}\left[{n+1 \atop 2}\right]$$

where $\left[{n \atop k}\right]$ denotes the Stirling Cycle number (Stirling number of the first kind) which is equal to the number of permutations of n objects that have exactly k cycles ([4], p. 261).

In the early 1980's, I became interested in the related expression,

$$S_n = \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{\ddots + \cfrac{1}{n}}}}}$$

also written as

$$S_n = \frac{1}{1+}\;\frac{1}{2+}\;\frac{1}{3+}\;\cdots + \frac{1}{n}$$

which is a simple finite continued fraction with n terms. Specifically, I wanted to know if there is a closed form expression for S_n, and if not, can S_n be expressed in terms of H_n?